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A

TREATISE ON SPHERICS,

COMPRISING

THE ELEMENTS

OF

Spherical Geometry,

AND

Of PLANE and SPHERICAL

Trigonometry,

TOGETHER WITH

A SERIES OF TRIGONOMETRICAL TABLES.

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CAMBRIDGE:

Printed by J. Smith, Printer to the University;

PUBLISHED BY J. MAWMAN, No. 39, LUDGATE STREET;

AND SOLD ALSO BY DEIGHTON AND SONS, CAMBRIDGE.

1816.

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PREFACE.

It is remarkable, that, since the days of Theodosius and Menelaus, scarcely have any books been written, professedly, on Spherical Geometry: and, that those authors who have entered upon it, as a collateral subject, have given its rudiments always very imperfectly, and often very inaccurately. More than one cause may, indeed, be assigned, in order to account for the neglect, into which this branch of Geometry seems to have fallen. But whether it has been deservedly so neglected, may well be doubted.

And, first, as a matter of mere speculation, interesting only to our curiosity, the comparison of Plane, with Spherical Geometry, is of itself a most fertile and ample field, for the exercise of mathematical research. Of the three great bases of reasoning, in the former branch, namely, the equality of the three angles, of every triangle, to two right angles, the properties of parallel lines, and the pro-

portionality of the sides about the equal angles of equiangular triangles, the first, it is soon discovered, has no existence at all in Spherics; the second, it is manifest, admits only of a partial application there, in the way rather of analogy, than of strict correspondence; and the third is found not to obtain in the case of any two triangles, on the same sphere, that are of different magnitudes. It might, therefore, on a first view of the question, seem unlikely that these two great provinces of Geometry should have many points of absolute contact, or near approximation. We are instantly prompted to enquire, under what circumstances spherical triangles are equal to one another; whether the relative position of the greater side, and the greater angle, be the same in these, as it is in plane triangles; whether, since the sides of equiangular triangles, on *equal* spheres, are equal, each to each, the sides of equiangular triangles, on *unequal* spheres, be not still proportionals, although they are not equal to one another: whether spherical triangles, on the same sphere, and on the same base, or on equal bases, and between *any* two parallel circles whatever of the sphere, be equal; and, if not, whether there is not *one* particular pair of parallel circles, between which this equality may subsist: whether, again, a great circle can be drawn, cutting two given great circles, so

as to make the alternate spherical angles equal; and lastly, whether there be any common principles in what relates to the contact of circles, in Plane, and in Spherical, Geometry.

The result of such enquiries is, that, in many cases, the same general enunciation equally applies to the propositions of the one and of the other class; that between many theorems, of the one and of the other, which cannot entirely correspond, there exists a very close analogy: and that, even the very dissimilarities of these two branches of mathematical science are no less worthy of notice, than their many examples of coincidence.

All this, however, is pure theory. It will also, readily be allowed, that the principal use of Spherical Geometry is to be looked for in the doctrine of Spherical Trigonometry: and it must, further, be admitted, that it is possible to deduce the solution of spherical triangles from one general theorem, which in order to be understood, requires very little knowledge of what is called Spherical Geometry. Any extensive study of that subject might, therefore, appear to be a needless labour. But it may be urged, on the other hand, that trigonometrical propositions can, in some instances, more easily be deduced from very simple constructions, than from the general theorem itself; that all,

which relates to the *species* of the sides and angles of spherical triangles, is best demonstrated by the geometrical method; and, generally, that, unless the student be well versed in the Geometry of the Sphere, he will not be able thoroughly to understand Spherical Trigonometry; much less will he be properly qualified to enter upon the task of learning Astronomy.

The greater number of authors, accordingly, who have written upon either of the two last mentioned subjects, have felt the necessity of supplying certain subsidiary geometrical truths. But this incidental kind of supply has commonly been insufficient; and some degree of confusion has, perhaps, been created, by the intermixture of propositions belonging to two different departments of the mathematics.

The principal design, therefore, of the following Treatise, is to separate Spherical Geometry and Spherical Trigonometry by a marked line of boundary; to investigate more fully than is usual, the former subject, and thus to make it a step, which may lead to the more perfect comprehension of the latter.

Of the method which has been pursued in the

execution of this design, some account remains now to be given.

The reader is supposed, in the first place, to be well acquainted with Euclid's Elements, or at least to have that book at hand. So that, in the following pages, after certain fundamental principles have been established, whenever, by the direct application of them, the proof of a spherical proposition becomes identified, in all its steps, with that of a corresponding proposition in Euclid, it has been thought sufficient to refer to Euclid's demonstration; in order to avoid all needless prolixity.

The partition, and the arrangement, of the materials have been regulated entirely by considerations of convenience, without any regard to precedent, even where it was to be found.

The definitions, belonging to the subject are introduced as they are wanted; instead of being prefixt, as they more commonly are, to the Part, or Section, in which they are used. Much might be said in favour of a general adoption of this method of proceeding. But, in Spherics, many of the definitions, which it is convenient to lay down, are founded on theorems: it would, therefore, have been preposterous to have announced them, before the propositions, upon which they are dependent,

had been proved. The definitions themselves have been very differently enunciated by different authors. Here, therefore, it became necessary to make a choice. A sphere, for example, has been defined, in the manner of Theodosius, rather than in that of Euclid; and a spherical angle, instead of being called the *Inclination of two Planes,* has been described, in language accommodated to the idea which we form of it, from sight; it being, in reality, the mutual inclination, not of two planes, but of two curve lines, on a convex surface.

But, the object chiefly aimed at, in the former part of the Treatise, was first to collect, and to confine within a very small number, such propositions as require any *dissection* of the sphere; and then to reduce all the rest to a Geometry, really practicable, on any spherical *surface,* and depending solely on the use of the rule and the compass. This, however, will hardly be understood, without some further explanation.

All the ancient writers, then, it is well known, and perhaps all the modern writers, on Spherics, in the construction of their problems, as well as in the proofs of their theorems, have recourse to certain graphical operations, which, are to be performed *in the interior* of the solid sphere. Now,

although in establishing the truth of a *theorem*, there seems to be no impropriety, in supposing such operations to have been executed, whenever they are evidently or demonstrably possible, and although, in demonstrating spherical theorems, there is often a kind of necessity for making this supposition, yet in the management of *problems*, the case is far from being the same. In this latter instance, a construction, which is really impracticable, can serve only, at most, to shew, what is seldom doubtful, — that the problem, which it pretends to solve, involves no impossibility. In reality, the problems, which are thus treated, either admit of a different and a practicable solution, or else they are totally useless.

To find the center of a given sphere, for example, is the first problem in Theodosius, and it has been copied by many succeeding writers. To effect what is thus proposed, the sphere is first to be cut by a plane; a perpendicular is next to be drawn, *within the sphere*, from the center of the section, and to be produced, both ways, until it meet the sphere's surface in two points: lastly, that straight line, so terminated, is to be bisected. Now this process is plainly impracticable, without wholly destroying the form of the given solid.

But neither is it neeessary, for the solution of any important spherical problem, to have actually found the sphere's center: the existence of such a point is established by a definition: and in all theorems, relating to a given sphere, it may fairly be considered as given. If, however, a sphere be put into the hands of a geometrician, he ought, without mutilating its figure, to be able to find the exact length of its diameter, and the exact places of the poles of any circles, the circumferences of which are in its surface: and all this he can do, by a series of common and easy operations, performed by a rule and compass; the whole solution of the problem, mainly depending upon this plain theorem, that if circles be described about triangles, which have the sides of the one severally equal to the sides of the other, the diameters of the circles are equal to one another.

In the following pages, therefore, regard has been had to *practicability*, in the solution of this, and of all other spherical problems. In the treatment of them, nothing has been directed, but what can easily be performed, either on the sphere's surface, or else on a given plane.

In the former part of the Treatise, not only the plan, but a considerable portion of the matter also,

is original, at least, if it be not new. Here, therefore, it would be great presumption, the nature of the subject being considered, to suppose that no mistakes have been committed. If the genius of Copernicus failed to guard him from errors, in this particular walk of science, men of inferior powers can hardly expect to have been more fortunate; although they may, perhaps, be allowed to hope for indulgence and excuse. In the latter part of the work, a more beaten track has been pursued. Care, however, has been taken, to present to the reader the subject of Trigonometry, in the two different points of view, from which it ought always to be considered. The learner ought always to be apprized, that the great problem, in which Trigonometry, properly so called, is comprised, may be solved in two very different ways: it is important, that he compare the two methods with one another, in order fully to apprehend the nature of the advantage, which the Algebraical method has, over the Geometrical: and, whilst he is taught Trigonometry, according to the former method, his attention should be constantly directed to the several bearings of the principal problem, in all its cases, and to the suppositions, which are often tacitly made, in its solution: in a word, he should be instructed to keep in view, the absolute dependence, which that solution has, upon the nu-

merical values of sines, and tangents, having been previously calculated; and he should be shewn the mode, or at least the possibility, of actually calculating such values, to any required degree of exactness.

These latter remarks apply, more especially, to the doctrine of Plane Trigonometry, which is here treated of only in a cursory manner: and it will be found, that they have not been lost sight of, in the short summary, which is given of that subject.

The great practical objects of Trigonometry, are undoubtedly best attained by computation: and when a proposition is to be investigated, which is afterwards to be employed in that way, the investigation itself, generally speaking, can best be conducted in the appropriate language, and by the peculiar processes of universal arithmetic. Whenever, therefore, actual calculation is the end ultimately proposed, recourse may be had without scruple, to the application of Algebra to Geometry: and, it seems, accordingly, as if no doubt could well be entertained, in the management of our subject, where Geometry should end, and where Algebra should begin.

No examples of calculation have been exhibited

in the following Treatise. These are to be sought for, in the introductions, which are commonly prefixt to books of logarithms. The doctrine of the corresponding variations of the sides and angles of triangles, and that of the projections of the sphere, have, also, been omitted: because the former, of those two subjects, seems more properly to belong to the method of fluxions, or to the differential method; and the latter, if it had been strictly and fully investigated, would have furnished matter sufficient for a separate Tract.

To the Vice-Chancellor, and the rest of the Syndics of the University Press, the Author begs leave to offer his sincere thanks, for the aid which they have afforded him, in his present undertaking: the greater part of the expence attending it, having been defrayed from the funds, which are at their disposal. He has also another debt, of another kind, to acknowledge; of which he can only acquit himself, by subjoining the following list of Authors, who have been consulted, with more or less advantage, in the course of his Work, —— Theodosius, Menelaus, Ptolemy, Muller, Copernicus, Clavius, Euler, Bertand, Cagnoli, Legendre, Lagrange, Delambre.

Trin. Coll.
Oct. 21, 1816.

ERRATA.

P. 9. l. 5. from the bottom, for *a* sphere's read *the* sphere's.
13. l. 9. from the bottom, for *though* read *through*.
17. Note, for 30 read 29.
19. l. 13. from the bottom, for *Cor.* read *Cor.* 1.
42. In the figure, the straight lines *EG* and *EF* are wanting.
139. In the last line, place a comma after *sphere*.
169. l. 1. dele the comma after *constructed*.
175. l. 6. from the bottom, dele *therefore*.
176. l. 14. from the top, place a comma after *sines*.
180. l. 8. from the top, at the beginning insert (25).
185. l. 4. from the bottom after *triangle*, read *L being the perpendicular as in Art.* 24, *then* (23) $\sin A = \frac{L}{S'}$; $\sin A' = \frac{L}{S}$; $\therefore \frac{S+S'}{S \backsim S'} =$ &c.
188. l. 5. from the bottom, after *co-tangent* put a comma.
225. l. 3. from the top, in the denominator of the fraction at the end of the line, for B' read B.
238. l. 8. from the top, for $\tan \frac{1}{2} (S+) S''$ read $\tan \frac{1}{2} (S+S'')$
—— l. 12. from the top, dele *less*.
253. l. 2. from the top, for *toublesome* read *troublesome*.
256. l. 7. from the top, for *Manduit* read *Mauduit*.
257. l. 4. in the note, before the word *another* insert the word *one*.
259. l. 1. for $\tan \frac{1}{2} (A \backsim A)$ read $\tan \frac{1}{2} (A \backsim A')$.

A

TREATISE

ON

Spherics.

PART I.

SPHERICAL GEOMETRY.

"Nullum autem est dubium, quin, si symptomata linearum curvarum, in superficie spherica descriptarum, eodem modo evolverentur, ac curvarum, in plano descriptarum, affectiones explicitæ fuerunt, nova Geometriæ pars prodiret, quæ non solum insigni varietate, sed elegantia quoque, inventorum se commendaret."——LEXELL.

(5.) A straight line drawn from a sphere's cer to its surface, is called a ***Radius of the Sphere.***

PROP. I.

(6.) *Theorem.* The common section of a sph and a plane is a circle.

First, if the cutting plane pass through the sphe center, it is evident (Art. 1, 2.) that the common sect is bounded by a line, of which every point is eq distant from that center; therefore, (E. Def. 15. 1.) common section is a circle.

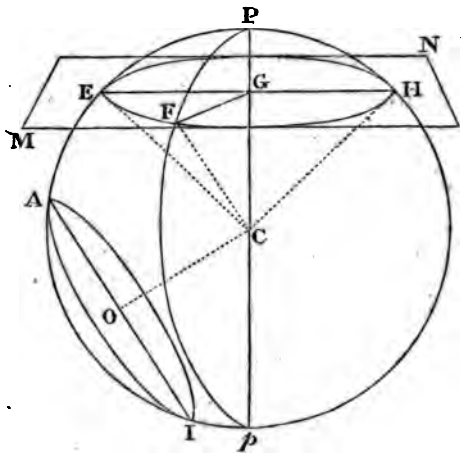
But, secondly, let *ApHE* be a sphere, of which is the center; let it be cut by a plane *MN*, wh does not pass through its center; and let *EFH* be common section of the plane, and the sphere: *EFH* a circle.

For, from *C*, let *CG* be supposed to be dra (E*. 11. 11.) at right angles to the plane *EFH*; ai in that plane, let there be drawn, from *G*, any t straight lines *GE* and *GF*, to the sphere's surfa Then (E. Def. 3. 11.) *CG* is at right angles to *GE* a *GF*; let, also, the points *C*, *E*, and *C*, *F* be joined.

Then, in the two right-angled triangles *CGE*, *CG*

* The letter E refers to Euclid's Elements; and, of the subseq numbers, the former denotes the Proposition, and the latter the Bc which are intended to be cited.

because the hypotenuse *CE* is equal to the hypotenuse *CF*, (Art. 1.), and the side *CG* is common to both the triangles, therefore (Demonstration of E. 14. 3.) the



third side *GE*, in the one, is equal to the third side *GF* in the other:

In the same manner may any other straight line as *GH*, drawn in the plane *EFH* from *G* to the sphere's surface, be shewn to be equal to *GE*, or *GF*. Therefore (E. Def. 15. 1.) the figure *EFH* is a circle.

The proposition may also be proved, by shewing that any two straight lines in *EFH*, drawn through *G*, and terminated by the sphere's surface, being chords of two equal circular sections of the sphere made by planes passing through them and the sphere's center, and being equally distant from the common center, are (E. 14. 3.) equal to one another; and that each of them being (E. 3. 3.) bisected, in *G*, by the perpendicular *CG*,

therefore, the straight lines GE and GF are equal to one another.

(7.) Cor. 1. If a sphere be cut by any two planes, passing through its center, the two circular sections, having a diameter of the sphere for their common diameter, shall bisect one another: they shall also, because their diameters are equal, be equal to one another*: all such sections of the sphere are, therefore, equal; they all bisect each other; and the center of the sphere is their common center.

(8.) Cor. 2. It has appeared, from the demonstration of Art. 6, that the center G of the circular section, EFH, of a sphere, made by a plane which does not pass through the sphere's center, is in the perpendicular CG, drawn to that section from the sphere's center C: and, conversely, (E. 13. 11.) the center C of the sphere is in the straight line GC drawn at right angles to the plane of the section EFH, from its center G. Therefore CG, the straight line joining the centers of the sphere and of any such section, is perpendicular to the plane EFH of that section: and, a plane which passes through the center of such a section and cuts it perpendicularly, passes also (E. Def. 4. 11 and 13. 11.) through the center of the sphere.

* Two circles, of which the diameters are equal, may be shewn to be equal, by applying the one to the other so that their centers shall coincide: for then the one figure must wholly coincide with the other. In the same manner, also, it may be shewn, that of two circles that which has the greater diameter is the greater.

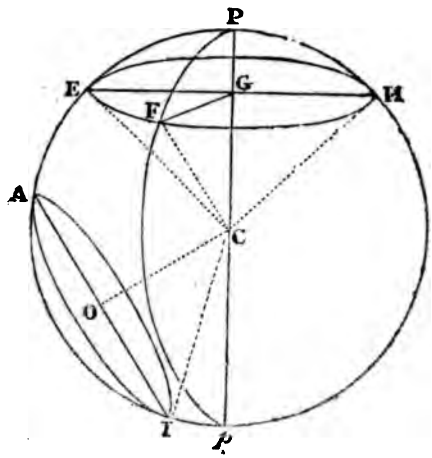
(9.) Def. Circles in a sphere are said to be equally distant from the sphere's center, when the perpendiculars drawn to their planes from the sphere's center are equal: and that circle is said to be the farther distant, upon the plane of which the greater perpendicular falls.

Prop. II.

(10.) *Theorem.* Of circles in a sphere, the greatest are those which pass through the sphere's center: of the rest, those are equal, which are equally distant from that center; and those which are nearer to the center, are greater than those which are more remote.

Let *PFp* be a circle passing through the center *C* of the sphere *AD*, and let *EH* and *AI* be circles in the sphere, which do not pass through its center: *Pp* is greater than *EH*, or than *AI*: If *EH* and *AI* be equidistant from the sphere's center, they are equal: If they be not equidistant from it, that, which is the nearer to it, is greater than that which is more remote.

For, let *G* and *O* be the centers of *EH* and *AI*; join



.*C*, *G* and *C*, *E* and *C*, *O* and *C*, *I*; join also *G*, *E* and *O*, *I*. Then (Art. 8.) the angles *CGE* and *COI* are right angles; wherefore (E. 16. and 19. 1.) *CE* is greater than *GE*, and *CI* than *OI*; that is, (Art. 7.) the radius of the circle *Pp* is greater than the radius of either of the other circles *EH*, or *AI*; therefore the circle *PFp* is greater than either *EH*, or *AI*.

Again, since (Art. 1.) *CE* is equal to *CI*, and since the angles at *G* and *O* are right angles, the squares of *CG* and *GE* are (E. 47. 1.) together equal to the squares of *CO* and *OI*; it is evident, therefore, that, if *CG* be equal to *CO*, that is, if the circles *EH* and *AI* be equidistant from the center *C*, the radius *GE* will be equal to the radius *OI*, and therefore the circle *EH* will be equal to *AI*: but, if *CG* be less than *CO*, that is, if the circle *EH* be nearer than *AI* to *C*, then the radius *GE* must be greater than the radius *OI*; otherwise the squares of *CG*, *GE* would not be equal to the squares of *CO*, *OI*: in this case, therefore, the circle *EH* is greater than *AI*.

(11.) Cor. 1. In the same manner, it may be shewn, conversely, that the greater of two circles, in a sphere, is nearer to the sphere's center than the less, and that equal circles, which do not pass through the sphere's center, are equally distant from it.

(12.) Cor. 2. Two equal circles in a sphere, which do not pass through the sphere's center, cannot have a common diameter.

For, then, they would have a common center; and the straight line, joining that center and the center of the sphere, would (Art. 8.) be perpendicular to the planes of both the circles; which (E. 5. 11.) is absurd.

(13.) Cor. 3. Two equal circles in a sphere which cut each other, but do not pass through the sphere's center, cannot have their centers both on the same side of that plane, which passes through their common section, and through the center of the sphere.

For, then, the perpendicular let fall from the sphere's center upon one of the circles must necessarily cut the other circle, and must consequently be greater than the perpendicular let fall on that other circle, from the sphere's center: the one of the two circles would therefore, (Art. 10.) be greater than the other: which is contrary to the supposition.

(14.) Cor. 4. Hence, no more than two circles, that are equal to one another, and do not pass through the sphere's center, can have a common section.

(15.) Def. Circles, in a sphere, the planes of which pass through a sphere's center, are called *Great Circles* of the sphere.

And *Lesser Circles* of a sphere are those, the planes of which do not pass through the sphere's center.

(16.) Cor. 1. A great circle may pass (Art. 6. and

E. 2. 11.) through any two given points on a sphere's surface: but unless the two given points be the extremities of a diameter of the sphere, there cannot be more than one great circle passing through them.

(17.) Cor. 2. Any two arches of equal circles, in a sphere, that have equal chords, are equal; and conversely: also the arches of great circles, which join the extremities of two equal arches of equal lesser circles, in a sphere, are equal to one another: and conversely, the arches of equal lesser circles, in a sphere, which join the extremities of two equal arches of great circles, are equal to one another. (Art. 7. and E. 28. and 29. 3.).

(18.) Cor. 3. Through any given point, on a sphere's surface, there may pass (E. 11. 11. and 18. 11.) a great circle, having its plane perpendicular to any given circle of the sphere: but (E. 13. 11. and Art. 15. and E. 2. 11.) through any point in the circumference of a given circle, there can only pass one great circle at right angles to the given circle.

(19.) Cor. 4. Two great circles of a sphere cannot (Art. 7.) be parallel: but there may be an indefinite number of lesser circles of the sphere (E. 14. 11.) all parallel to any great circle, and to one another.

(20.) Def. The *Axis* of any circle, in a sphere, is a diameter of the sphere perpendicular to the plane of that circle.

(21.) Def. And the two extremities of the axis

of a circle, in a sphere, are called the *Poles* of that circle.

(22.) Cor. 1. The center of any given circle in a sphere, the two poles of that circle, and the sphere's center, are in the same straight line, namely, the axis of the circle.

This is evident, if the given circle be a great circle, from Art. 1. 20. 21; and if the given circle be a lesser circle of the sphere, it follows from Art. 8. 20. 21.

(23.) Cor. 2. If the circumference of a great circle pass through one of the poles of a given circle in a sphere, it passes also through the other pole of that circle.

For the plane of the great circle passes through the sphere's center; if, therefore, it pass through one of the poles of the given circle, it must, also, (Art. 22.) pass through the other pole.

(24.) Cor. 3. All the circles in a sphere, which have the same axis, have the same poles (Art. 21.): and all the circles in a sphere, which have the same poles, have the same axis: otherwise, two straight lines would inclose a space.

(25.) Cor. 4. All the circles in a sphere, which have the same axis, that is, which have the same poles, are (Art. 20. and E. 14. 11.) parallel to one another: and circles in a sphere, which are not parallel, have not the same poles:

The truth of the converse proposition also follows, from the converse of E. 14. 11; which is easily proved, *ex absurdo*:

And, therefore, if two great circles be perpendicular to one another, either of them is (E. 18. 11.) perpendicular to all the parallels of the other.

Prop. III.

(26.) *Theorem.* There cannot be more circles in a sphere, all equal and parallel to one another, than two.

For (Art. 25.) all such circles will have the same axis, and their centers, which (Art. 22.) are all in that common axis, are equally distant (Art. 10.) from the sphere's center, which is a point (Art. 20.) in that axis: and it is evident, that there cannot be more than two points, in a straight line, all equally distant from any, the same, given point in that line.

Prop. IV.

(27.) *Theorem.* If two great circles, in a sphere, be perpendicular to one another, the poles of either of them are in the other's circumference: and, conversely, if the poles of the one great circle be in the circumference of the other, the poles of the latter are also in the circumference of the former, and the two circles are perpendicular to one another.

First, if the two great circles cut each other at right angles, a diameter in either of them, drawn perpendicular

to their common section, is (E. Def. 4. 11.) perpendicular to the other circle; and therefore (Art. 20. 21.) its extremities are the poles of that other circle.

Again, if the poles of the one great circle be in the circumference of the other, the axis (Art. 20.) of the one lies in the plane of the other; and therefore (Art. 20. and E. 18. 11.) the plane of the one is perpendicular to the plane of the other: therefore (Art. 20. and E. Def. 4. 11. and 13. 11.) the poles of each circle are in the other's circumference.

Prop. V.

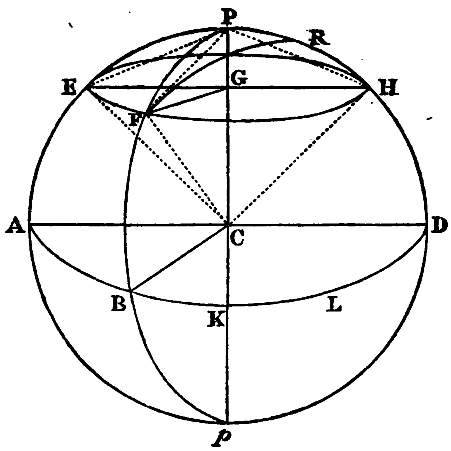
(28.) *Theorem.* If a great circle pass through the diameter of a lesser circle, in a sphere, it cuts the lesser circle at right angles, and passes through its poles: and, conversely, if a great circle pass through the poles of a lesser circle in a sphere, it shall cut the lesser circle at right angles, and, pass though its diameter: and, also, if it be at right angles to the lesser circle, it shall pass through its diameter, and its poles.

Let EH be the common section of a great circle $PEpH$, in the sphere $PEpH$, and of a lesser circle EFH: and first let EH be a diameter of EFH; then the great circle $PEpH$, cuts EFH at right angles, and passes through its poles.

For, let C be the sphere's center; and let the point G,

in *EH*, be the center of the circle *EFH*, and let *C*, *G* be joined:

Then (Art. 8.) the straight line *CG* is perpendicular to the circle *EFH*: but the plane of the great circle



EPH, passes, by the hypothesis, through *G*, and it also (Art. 15.) passes through *C*; it passes, therefore, through the straight line *CG*: therefore, (E. 18. 11.) the great circle *EPH* is perpendicular to the lesser circle *EFH*: and *PEH* has been shewn to pass through *CG*, which line produced is (Art. 20.) the axis of *EFH*: wherefore (Art. 21.) it passes through the poles of *EFH*.

Conversely, let the great circle *PEpH* pass through the two poles *P* and *p* of *EFH*, and let the straight line *EH* be the common section of the circles: then *PEpH*

cuts *EFH* at right angles, and *EH* is a diameter of *EFH.*

For since *PEpH* passes through the poles of *EFH,* it passes through the axis; and, consequently, (E. 18. 11.) it is perpendicular to *EFH.* Again, since *PEH* passes through the axis of *EFH,* and that (Art. 22.) the center of *EFH* is in its axis; therefore *PEH* passes through the center of *EFH,* and the common section *EH* is, therefore, a diameter of *EFH.*

Lastly, let *EPH* be perpendicular to *EFH,* and let *EH* be the common section of the planes: *EH* is a diameter of *EFH.*

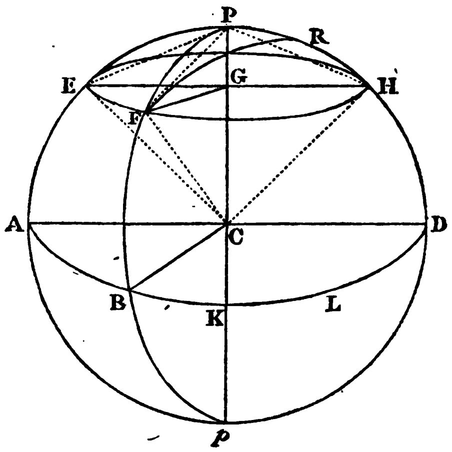
For, since *PEH* is a great circle, the sphere's center *C* is (Art. 15.) in its plane: from *C,* in the plane of *PEH,* let there be drawn *CG* perpendicular to the straight line *EH,* which is the common section of the two circles; and from *G,* draw, in the plane of the circle *EFH,* any straight line *GF* to the circumference: then, the angles *CGE, CGF* and *CGH* are right angles, (E. Def. 4. and hypothesis); and it may be shewn, as in Art. 6. that *GE, GF* and *GH* are equal; therefore (E. 9. 3.) *G* is the center; and *EGH* is a diameter of *EFH*; therefore, by the first part of the proposition, *PEH* passes through the poles of *EFH.*

Prop. VI.

(29.) *Theorem.* If any two great circles in a sphere be

each perpendicular to a given circle, the intersections of their circumferences shall be the poles of the given circle.

If the given circle be a great circle, the truth of the proposition is manifest, from Art. 27. But let the two



great circles *PEp, PFp,* of which the circumferences cut one another in *P* and *p,* be each of them perpendicular to the lesser circle *EFH*: *P* and *p* are the poles of *EFH.* For, join *P, p* : the straight line *Pp,* which (Art. 7.) passes through the sphere's center, being the common section of the circles *PEp, PFp,* is (E. 19. 11.) perpendicular to the circle *EFH*: therefore (Art. 20.) *Pp* is the axis, and (Art. 21.) *P* and *p* are the poles of *EFH.*

(30.) Cor. Through any point, in a sphere's surface, which is not the pole of a given circle, there

cannot pass more than one great circle perpendicular to the given circle.

PROP. VII.

(31.) *Theorem.* Any two points, in the circumference of any, the same circle, in a sphere, are equally distant from either of that circle's poles: whether their distances, therefrom, be measured, on the sphere's surface, by arches of great circles passing through those points and the pole; or by straight lines, joining those points and the pole: and conversely, if a point on a sphere's surface be equally distant from all the points of the circumference of a circle in the sphere, that point is the pole of the circle.

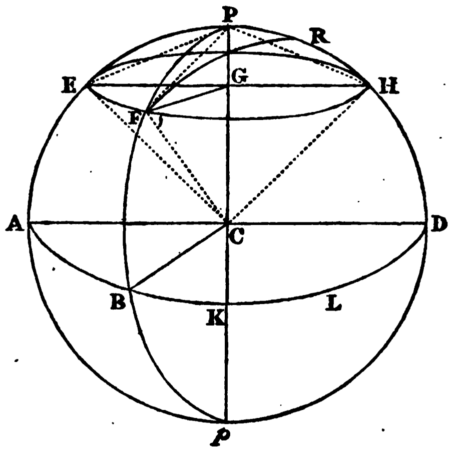
Let *F* and *H** be any two points, in the circumference of the circle *EFH*, of which *P* is either of the poles; and let *PF* and *PH* be two arches of great circles joining *P*, *F* and *P*, *H*: the arch *PF* is equal to the arch *PH*: also, the straight line joining *P*, *F*, is equal to the straight line joining *P*, *H*.

For, join *C*, *P*; and let *CP* cut the circle *EFH* in *G*: then (Art. 22.) *G* is the center of *EFH*: join *G*, *F* and *G*, *H*.

And, because *GF* is equal (E. Def. 15. 1.) to *GH*, and *CF* equal (Art. 1.) to *CH*, and *CG* common to the two triangles *CGF*, *CGH*, therefore (E. 8. 1.) the

* See the figure of Art. 30.

angle *GCF* is equal to *GCH*: but (Art. 15. and 7.) *PF* and *PH* belong to equal circles: wherefore, (E. 26. 3.) the arch *PF* is equal to the arch *PH*: also (E. 29. 3.) the chord of *PF* is equal to the chord of *PH*.



And, in the same manner, any other point, in the circumference of *EFH*, may be shewn to be equally distant from either of its poles: all the points, therefore, in the circumference of a circle, in a sphere, are equidistant from either of that circle's poles.

Conversely, let the point *P* be equally distant from all the points of the circumference of the circle *EFH*: then, is *P* the pole of *EFH*.

For, let *F* and *H* be any two points in the circumference of *EFH*; let *C* be the sphere's center; join *C*, *P* and let *CP* cut the circle *EFH* in *G*: join also *P*, *F* and *P*, *H*, and *G*, *F*, and *G*, *H*, and *C*, *F*, and *C*, *H*: then (Art. 1.) *CF* is equal to *CH*: and because, by the

hypothesis, the arch *PF* is equal to the arch *PH*, therefore (E. 29. 3.) the chord *PF* is equal to the chord *PH*; and two sides of the triangle *PFC* are equal to two sides of *PHC*, and a third side *CP* is common to both the triangles: wherefore, the angle *FCP* is equal to *HCP* (E. 8. 1.): again, since the angle *FCG* is equal to *HCG*, and that *FC*, *CG* are equal to *HC*, *CG*, in the triangles *GFC*, *GHC*, therefore (E. 4. 1.) *GF* is equal to *GH*. And, if *E* be any other point in the circumference of the circle *EFH*, and *G*, *E* be joined, it may be shewn in the same manner, that *GE* is equal to *GF*, or to *GH*. Therefore (E. 9. 3.) *G* is the center, and (Art. 22.) *CGP* is the axis, and *P* the pole, of the circle *EFH*.

(32.) Cor. If any number of great circles pass through the common poles of two parallel circles, in a sphere, the arches of the great circles, intercepted between the parallels, shall be equal to one another.

(33.) Cor. 2. The pole of any given circle, in a sphere, bisects the arch of a great circle subtended by the diameter of the given circle (Art. 20. 21. and E. 30. 3.).

(34.) Def. Any two points having been taken on the surface of a sphere, their *Direct Distance* is the straight line which joins them: and their *Spherical Distance* is the arch of a great circle which joins them:

Also, the distance of the pole of any circle, in a sphere,

from any point in the circumference of the circle, measured by an arch of a great circle of the sphere, is called the *Polar Distance* of that circle.

(35.) Def. The fourth part of the circumference of a great circle, in a sphere, is called a *Quadrant*.

Prop. VIII.

(36.) *Theorem.* The polar distance of a great circle, in a sphere, is a quadrant; and its chord is equal to the side of a square inscribed in a great circle: And if the polar distance of a given circle in a sphere be a quadrant, the circle is a great circle.

For first, since the axis is (Art. 20.) perpendicular to the plane of the great circle, it is manifest that the polar distance subtends a right angle: therefore (E. 33. 6.) it is a quadrant: and (E. 4. 4.) its chord is equal to the side of a square; inscribed in a great circle of the sphere.

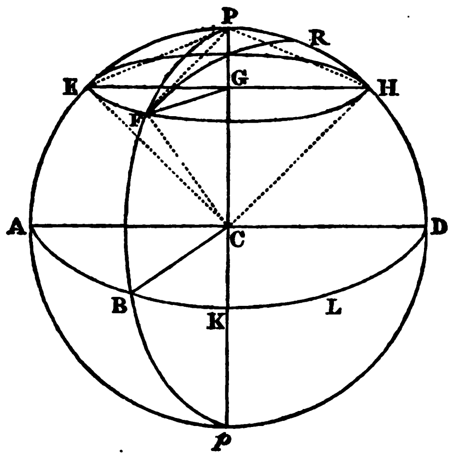
Secondly, if the polar distance, which (Art. 33.) is the same at all points of the circumference of the circle, be a quadrant, it is evident that any diameter of the given circle will bisect the great circle, which passes both through its axis, and through that diameter: It must, therefore, itself be a great circle of the sphere.

Prop. IX.

(37.) *Theorem.* If a point, on a sphere's surface, be at the distance of a quadrant from more than one point

in the circumference of a given great circle of the sphere, it is a pole of that great circle.

Let the point *P*, on the surface of the sphere *PApD*, of which the center is *C*, be at the spherical distance of



a quadrant, from any two points, *A* and *B*, in the circumference of the great circle *ABD*, that is, let the arches of great circles *PA* and *PB* be quadrants: then is *P* the pole of *ABD*.

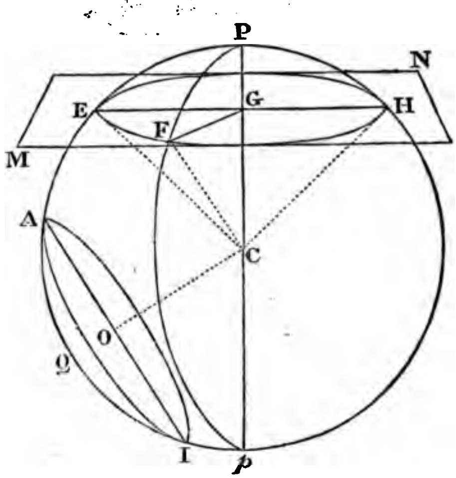
For, join *C, A* and *C, B*; and since *CA* and *CB* are in the plane of *ABD*, and that (E. 33. 6.) the angles *PCA, PCB* are right angles, therefore (E. 4. 11.) *PC* is at right angles to *ABD*; and consequently (Art. 20. 21.) *P* is the pole of the great circle *ABD*.

Prop. X.

(38.) *Theorem.* The polar distances of equal circles, in a sphere, are equal to one another: And circles in a sphere, which have equal polar distances, are equal to one another.

If the circles be great circles, the polar distances are equal, because (Art. 36.) each of them is a quadrant.

But, let *EH* and *AI* be two lesser circles in the sphere *EAH*; and, first, let the circle *EH* be equal to



AI: the polar distance of *EH* is equal to the polar distance of *AI*:

For, let *EH* and *AI* be any diameters of the circles *EH* and *AI*: and let *EPH* and *AQI* be arches of the great circles of the sphere which pass through the diameters *EH* and *AI* respectively. Then (Art. 29.) a pole of *EH* is in *EPH*, and a pole of *AI* is in *AQI*.

And, since (Art. 7.) all great circles, in a sphere, are equal to one another, and that the diameter *EH* is equal to the diameter *AI*, because, by the hypothesis, the circle *EH* is equal to the circle *AI*, therefore, (E. 28. 3.) the arch *EPH* is equal to the arch *AQI*: but (Art. 33.)

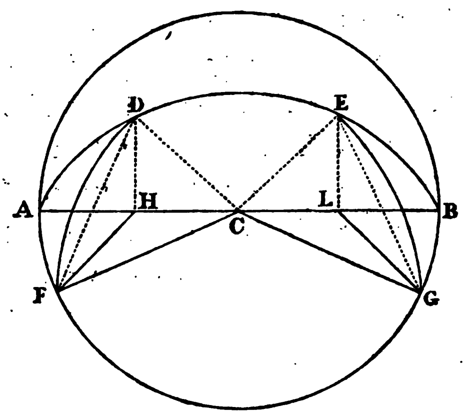
the poles of *EH* and *AI* are in the bisections of the arches *EPH* and *AQI*: since, therefore, the arches *EPH* and *AQI* are equal to one another, their halves, that is, the polar distances of the circles *EH* and *AI*, are equal to one another.

The converse proposition is proved, in like manner, by the help of E. 29. 3.

Prop. XI.

(39.) *Theorem.* If a great circle cut any other circle, in a sphere, at right angles, any two points, in the great circle's circumference, that are equally distant from the two extremities of the common section of the circles, are at equal distances from two points in the circumference of the other circle, that are also equally distant from the two extremities of the common section of the circles: and conversely.

Let the great circle *ADB* cut any other circle in the



same sphere, as *AFGB* at right angles, and let *AB* be

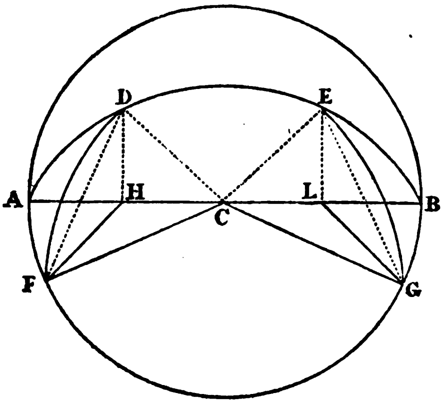
the common section of the circles: also, first, let the point *D*, in *ADEB*, be at the same distance from *A*, that *E* is from *B*; let the point *F* also, in *AFGB*, be at the same distance from *A*, that *G* is from *B*: then is *D* at the same distance from *F*, that *E* is from *G*. For (Art. 7. and 28.) *AB* is a diameter of *AFGB*: let, therefore, *C* be its center: suppose *DH* and *EL* to be drawn from the points *D* and *E*, in the plane of *ADB*, perpendicular to *AB*: wherefore (E. Def. 4. 11.) *DH* and *EL* are perpendicular to the plane *AFGB*: join *C*, *D*, and *C*, *F*, and *C*, *E* and *C*, *G*, and *D*, *F* and *H*, *F*, and *E*, *G* and *L*, *G*.

And since, by the hypothesis, the direct distance of *A*, *D* is equal to the direct distance of *E*, *B*, the arch *AD* (E. 28. 3.) is equal to *EB*: for the same reason, the arch *AF* is equal to *BG*: wherefore (E. 27. 3.) the angle *ACD* is equal to *BCE*, and the angle *ACF* to *BCG*: in the two right-angled triangles, therefore, *DHC*, *ELC*, the side *CD* is equal to *CE*, each being a radius of the same circle, and the angle *DCH* to *ECL*: therefore (E. 26. 1.) *DH* is equal to *EL*, and *CH* to *CL*: and, because *CH* is equal to *CL*, and the radius *CF* to the radius *CG*, and that the angle *HCF* has been proved to be equal to *LCG*, therefore (E. 4. 1.) *HF* is equal to *LG*; and *DH* has been shewn to be equal to *EL*, and the angles at *H* and *L* (E. Def. 3. 11.) are right angles: therefore (E. 4. 1.) *DF* is equal to *EG*.

Next, let *D* be at the same distance from *A* that *E*

is from *B*, and let *DF* be equal to *EG*: then, is the distance of *A*, *F* equal to the distance of *BG*.

For, the same construction having been made, it may be shewn, as before, that *CH* is equal to *CL*, and *DH* to *EL*; and that the angles *DHF*, *ELG* are right angles: and since, by the supposition, *DF* is equal to *EG*; it



follows from E. 47. 1. that *HF* is equal to *LG*: and the radius *CF* is equal to *CG*: therefore the three sides of the triangle *CHF* are equal to the three sides of *CLG*, each to each: wherefore (E. 8. 1.) the angle *HCF* is equal to the angle *LCG*, the arch *AF* (E. 26. 3.) to the arch *BG*, and also (E. 29. 3.) the direct distance of *A*, *F* to the direct distance of *B*, *G*.

(40.) Cor. It is manifest, from the demonstration, and from E. 28. 3. that if, instead of the direct distances of the several points, the circular arches intercepted between them, be substituted, the proposition is still true.

It is evident, also, that the very same proof may be applied to shew, that, if any other circle equal to $AFGB$, in the same sphere, or in an equal sphere, be cut at right angles by a great circle, and the points E and G be taken at the same distances from either extremity of the common section, as D and F, respectively, are from A, DF will still be equal to EF.

PART I.

THE ELEMENTS OF

Spherical Geometry.

SECTION II.

ON SPHERICAL ANGLES AND THEIR MEASURES.

DEFINITION.

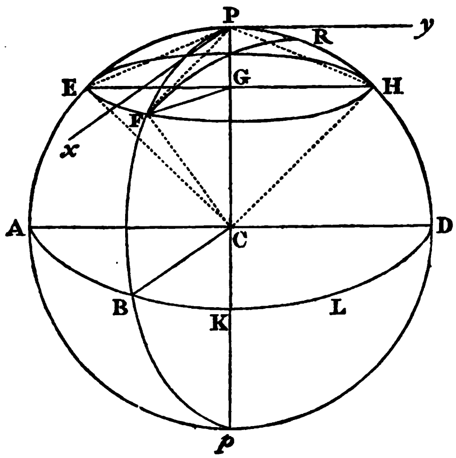
(41.) A *Spherical Angle** is the inclination of two circular arches to one another, in a sphere's surface, which meet together, but do not belong to the same circle:

If, therefore, two straight lines be drawn touching any two circular arches, on a sphere's surface, which include a spherical angle, the one straight line touching the one arch, and the other the other, in their point of concourse,

* The *reclination* of two such arches from one another, the measure of which might exceed two right angles, is designedly excluded from the definition of a spherical angle.

When an angle is called a Spherical Angle, without any further specification, it is to be understood that the angle is the inclination of two *great* circles to one another.

it is evident, that the inclination of these tangents, to one another, is the same as the inclination of the arches to one another. The plane rectilineal angle, therefore, contained by the tangent straight lines, may be taken as a measure of the spherical angle contained by the arches.



Thus, if PF and PH be any two circular arches, in the surface of the sphere PHp, which meet in P, and Px be a straight line touching PF in P, and Py a straight line that touches PH in P, the plane rectilineal angle xPy is equal to the spherical angle FPH.

(42.) COR. 1. If two arches of circles, in a sphere's surface, meet together, they make with one another either two right angles, or two angles that are, together, equal to two right angles (Art. 41. and E. 13. 1.)

(43.) COR. 2. If two arches of circles, in a sphere's surface, cut one another, the spherical opposite, or vertical, angles shall be equal (Art. 41. and E. 15. 1.)

(44.) Cor. 3. Hence, if two arches of circles cut one another, in a sphere's surface, the angles which they make, at the point where they cut, are, together, equal to four right angles (Art. 42.)

Prop. I.

(45.) *Theorem.* All the straight lines which touch any number of great circles, of a sphere, in their common point of intersection, are in the same plane.

For (E. 18. 3.) the tangent straight lines are all perpendicular to that diameter of the sphere, which is the common section of the great circles: wherefore (E. 5. 11.) they are all in the same plane.

(46.) Cor. It is manifest (E. 4. 11.) that a plane containing the two straight lines which touch two great circles, in a sphere, at their point of intersection, cannot meet the sphere, in any other point, than in that point of intersection: a plane, therefore, which meets a sphere in any point, so as to be at right angles to the sphere's radius, at that point, is said to touch the sphere in that same point.

Prop. II.

(47.) *Theorem.* All the spherical angles made by any number of arches of great circles, in a sphere, at the point where they cut, are, together, equal to four right angles.

For, since the tangents to the great circles, at the

point where they meet, are (Art. 45.) in the same plane, all the angles made by them, at their point of concourse, are (E. Cor. 2. 15. 1.) together equal to four right angles: wherefore (Art. 41.) all the spherical angles made by the arches of the great circles, at that point, are together equal to four right angles.

Prop. III.

(48.) *Theorem.* If, at a point in a circular arch, on the surface of a sphere, two arches of circles make the adjacent spherical angles, together, equal to two right angles, the two circular arches, shall be arches of one and the same circle.

The proposition is proved, by means of Art. 42, in the same manner as the fourteenth Proposition, in the first Book of Euclid's Elements.

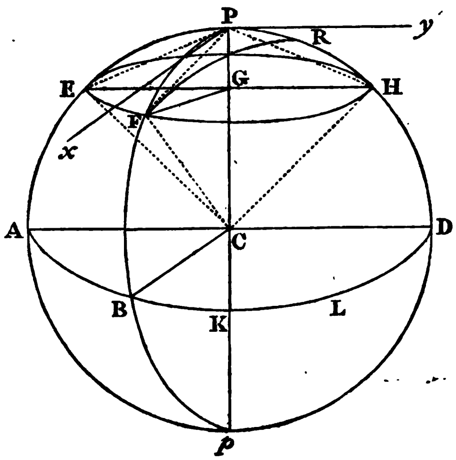
Prop. IV.

(49.) *Theorem.* If the planes of two circles in a sphere, of which one, at least, is a great circle, be perpendicular to one another, the circumferences of the circles shall cut one another at right angles:

And, conversely, if the circumferences of two circles in a sphere of which one, at least, is a great circle, cut one another at right angles, the planes of the circles shall be perpendicular to each other.

In the sphere $EPHp$, let PFp and EFH be two

circles of which *PFp* is a great circle: if the circles



are perpendicular to one another, they must (Art. 7. and 29.) cut each other: let, therefore, their circumferences cut one another in *F*; and, in the latter of the two cases, stated in the proposition, they cut one another, by the hypothesis: let, also, *FG* be the common section of their planes: and, first, let the plane of *PFp* be perpendicular to the plane of *EFH*: then the spherical angles, at *F*, are right angles.

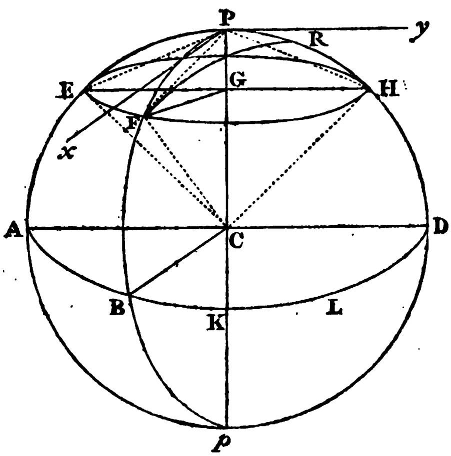
For, since the circles are perpendicular to one another, the poles, and also a diameter of *EFH* are (Art. 27. or 29.) in *PFp*: let *P* and *p* be the poles of *EFH*; and join *P*, *p*; *Pp* (Art. 22.) passes through the sphere's center, and also through *G*, the center of *EFH*; which point, (Art. 29.) if *EFH* be a lesser circle, is in the common section of the planes of the two circles. In the plane of *EFH*, let *GE* be drawn perpendicular to the

radius *GF*; wherefore (E. 18. 3. and 28. 1.) the straight line touching *EFH*, in *F*, is parallel to *EG*: but (E. Def. 4. 11.) *EG* is perpendicular to the plane *PFp*; therefore (E. 8. 11.) the tangent of *EFH*, at *F*, is perpendicular to the plane *PFp*, and, therefore, it is perpendicular (E. Def. 3. 11.) to the tangent of *PFp*, at *F*. Since, therefore, the two tangents, at *F*, are perpendicular to one another, the spherical angles at *F* (Art. 41.) are right angles.

Secondly, if the circumferences of the two circles cut one another at right angles in *F*, their planes are perpendicular to one another. For, let *G* be the center of *EFH*; and join *F*, *G*: then, if both the circles be great circles, since the tangent of *EFH*, at *F*, is (E. 18. 3.) perpendicular to *FG*, and is also (hypothesis and Art. 41.) perpendicular to the tangent of *PFp* at *F*, it will be perpendicular (E. 4. 11.) to the plane of *PFp*; and, therefore, the plane of *EFH*, since it passes through that tangent, is (E. 18. 11.) perpendicular to the plane of *PFp*. But let *EFH* be a lesser circle: then, if *PFp* pass through *FG*, it is (Art. 29.) perpendicular to the plane of *EFH*: and if *PFp* do not pass through *FG*, let the great circle *FR* pass through it: wherefore (Art. 29. and the first case of this proposition) the angle *RFE* is a right angle: and, by the hypothesis, *PFE* is a right angle: therefore the angle *RFE* is equal to *PFE*, the greater to the less; which is impossible.

(50.) Cor. 1. If the circumferences of two circles,

in a sphere, of which one, at least, is a great circle, cut one another at right angles, the circumference of the great circle (Art. 49. and Art. 27. and 29.) passes through the poles of the other: and, conversely, if the circumference of a great circle, in a sphere, pass through the poles of another circle, their circumferences (Art. 27. 29. 49.) shall cut one another at right angles: and when both the circles, are great circles, if the poles of the one be in the other, then the poles of either of them are in the other circle.



(51.) Cor. 2. Hence, through any point in a sphere's surface, which is not the pole of a given circle of that sphere, there cannot pass more than one arch of a great circle, perpendicular to the circumference of the given circle: and the pole of a given circle, in a sphere, is in the intersection of any two arches of great circles, which cut its circumference perpendicularly.

Prop. IV.

(52.) *Theorem.* A spherical angle is equal to the

inclination, toward each other, of the planes of the two great circles, by the circumferences of which it is contained.

Let the spherical angle *FPH** be contained by the arches *PF* and *PH*, of the two great circles *PFp*, *PHp*, the planes of which cut one another in the diameter *Pp*. The angle *FPH* is equal to the inclination of the plane *PFp* to the plane *PHp*.

For, let there be drawn the straight line *Px*, touching *PFp* in *P*, and *Py*, touching *PHp* in *P*: and from any point *G*, in *Pp*, let there be drawn *GF*, in the plane *PFp*, and *GH* in the plane *PHp*, each of them perpendicular to *Pp*: so that (E. Def. 6. 11.) the angle *FGH* is the mutual inclination of the two planes. Then (E. 18. 3. and 28. 1.) *GF* is parallel to *Px*, and *GH* to *Py*; therefore (E. 10. 11.) the angle *FGH* is equal to the angle *xPy*; that is, the inclination of the two planes *PFp*, and *PHp*, is equal to the angle *xPy*; which (Art. 41.) is the same as the spherical angle *FPH*.

(53.) Cor. Hence, if two concentric spherical surfaces be cut by any, the same, two planes, each passing through their common center, the spherical angle, contained by the arches, which are the common sections of the one spherical surface and the two planes, shall be equal to the spherical angle, contained by the common sections of the other spherical surface and the two planes†.

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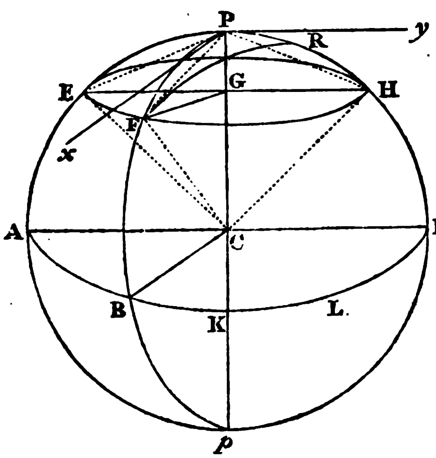
* See the figure in Art. 54.   † See the figure in Art. 119.

Prop. V.

(54.) *Theorem.* A spherical angle is measured by the arch of any circle, included between the arches of the great circles which contain that angle, and having its pole in the angular point.

Let EPF be a spherical angle, in the surface of a sphere, of which C is the center; and let PE and PF cut the circumference of any circle EFH, of which P is the pole, in the points E and F; the arch EF measures the spherical angle EPF.

For, join C, P; and, since P is the pole of EFH, CP (Art. 22.) cuts EFH perpendicularly, and passes through its center G: join G, E and G, F: where-



fore (E. Def. 3. 11.) the angles CGF and CGE are right angles. And (Art. 15.) CP is the common section of the planes PEp, and PFp: also (Art. 29.) GF and GE

are in those same planes: wherefore (E. Def. 6. or 4. 11. and Art. 49. and 52.) the angle EGF is equal to the spherical angle EPH. But (E. 33. 6.) the angle EGF has for its measure the arch EF: therefore, the arch EF measures also the spherical angle EPF.

(55.) Cor. 1. A spherical angle is to four right angles, as the arch which measures the spherical angle, is to the whole circumference, of which that arch is a part: and, therefore, spherical angles, in the same sphere, or in equal spheres, that are measured by equal arches of equal circles, are equal to one another: and, conversely.

(56.) Cor. 2. The semi-circumferences of two great circles, in a sphere, make with each other equal spherical angles, at their two points of intersection.

Prop. VI.

(57.) *Theorem.* The inclination, toward each other, of any two circles, in a sphere, which cut one another, is measured by the spherical distance of the pole of the one from the pole of the other: and, if the two circles, which cut one another, be great circles, of the sphere, the spherical distance of a pole of the one, from a pole of the other, measures also the spherical angle contained by their circumferences.

For, if two straight lines be drawn at right angles to the common section of the circles, one in the plane of each circle, from the middle point of that common section, they will pass through the centers of the two circles (E. 1. 3. Cor.), and will meet (Art. 20. 22.) their axes at right angles:

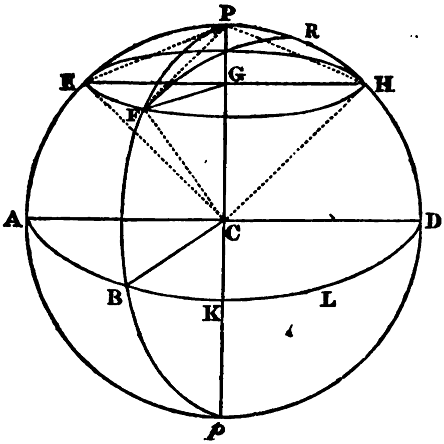
the common section, also, being at right angles to each of the straight lines so drawn, is at right angles to the plane (E. 4. 11.), which passes through them; therefore (E. 18. 11.) each of the circles is perpendicular to that plane, which passes through their centers, and consequently, (Art. 27. 28.) their axes are both of them in that plane; that is, the axes, and the two straight lines which meet them, drawn at right angles to the common section of the circles, from the middle of that section, are in the same plane.

It is manifest, therefore, from E. 32. 1. that the angle, contained by the axes, is equal to the angle contained by the two straight lines so drawn; which (E. Def. 6. 11.) is the inclination of the one circle to the other: but the angle, contained by the axes, is measured (Art. 21. and E. 33. 6.) by the spherical distance of the poles: therefore, the spherical distance of the poles measures the inclination of the one circle, to the other: and consequently, if the circles be great circles, it measures (Art. 52.) the spherical angle contained by their circumferences. Or, the latter part of the proposition may be proved in the following manner:

Let the spherical angle APB be contained by arches PA and PB of great circles, of which the poles are K and L; and let $LKBA$ be a great circle passing through K and L. The arch KL measures the spherical angle APB.

For (Art. 27.) the poles of the circle LBA are both

in *PAp* and in *PBp*; wherefore *P* and *p* are the poles of *LBA*; and (Art. 54.) *AB* measures the spherical angle



APB: but (Art. 36.) *AK*, and *BL* are quadrants; and, therefore, equal to one another: take from both the common part *BK*, and there remains *KL* equal to *AB*; and *AB* has been shewn to be a measure of the spherical angle *APB*: wherefore, also, *KL* measures the angle *APB**.

(58.) Cor. A great circle, which passes through the common section of two equal lesser circles, in a sphere, is inclined equally to the planes of them both: and if a great circle, also, be equally inclined to two other great circles, in a sphere, its circumference bisects, at right angles, the spherical distance of their poles.

* It is easily shewn, from Art. 50, that the angle contained by the circumferences of *any* two circles, in a sphere, is equal to the spherical angle subtended, by the distance between their poles, at the point in which the circles cut one another.

*Postulates**.

(59.) Let it be granted, that, if two points be given in a sphere's surface, their direct distance asunder may be measured.

Let it be granted, also, that, if two points on a sphere's surface be given, a circle may be described in the sphere, so that its circumference shall pass through one of the two given points, and have all its points equally distant from the other given point: and, that, therefore, a circle, in a given sphere, may be described, which shall have the direct distance, between any point in its circumference and its pole, equal to any given straight line that is less than a diameter of the sphere.

(60.) COR. Either of two given points, on a sphere's surface, is the pole of a circle described, in the manner specified in the latter postulate, so as to pass through the other given point (Art. 31.)

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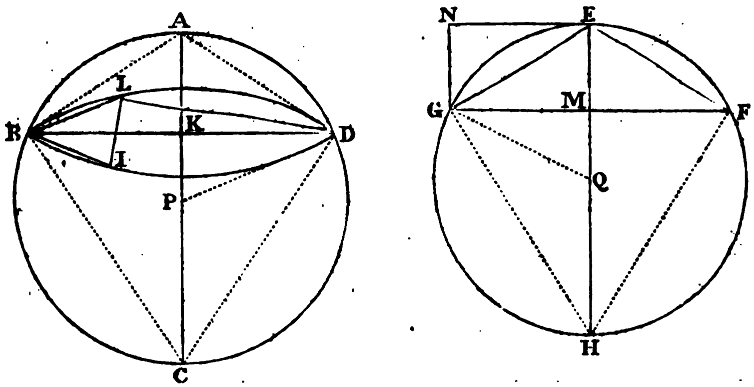
* The *nature* of the measures of spherical angles having been ascertained, the proposition, next in order, would have been the actual determination of an arch, which should measure a given spherical angle, on a given sphere; if that problem could have been solved by what has been premised. But, the passage from theory to practice cannot be effected, unless the possibility of certain graphical operations on the surface of a sphere, in addition to those of plane Geometry, be first granted. Hence arises the necessity of having recourse, in this place, to postulates. The operations, alluded to, may all be performed by an instrument called the *Spherical Compass*: and it is convenient, first, to apply them to the determination of the diameter of a given sphere, and the poles of a given circle, in a sphere, before the problem, which has been announced, be attempted.

Prop. VII.

(61.) *Problem.* To draw a straight line, that shall be equal to the diameter of a given circle, in a given sphere.

Let *BLD* be a given circle, in the given sphere



ABCD: it is required to draw a straight line, that shall be equal to the diameter of the circle *BLD*.

Let *BD* be the diameter of the given circle: take any three points in the circumference, as *I*, *B* and *L*: suppose the three points, *I*, *B* and *L*, to be joined by straight lines: then, (Art. 59.) the three chords *IB*, *BL*, *LI*, may be considered as given. Make, therefore, (E. 22. 1.) the triangle *EFG*, having its sides equal to the sides of *IBL*; namely, *EG* to *BL*, *EF* to *LI*, and *FG* to *IB*: draw (E. 11. 1.) *FH* and *GH*, at right angles to *EF* and *EG*, respectively, and let them meet in *H*: join *E*, *H* : *EH* is equal to *BD*.

For, let *D*, *L* be supposed to be joined: then, since the angles *EFH*, *EGH* are right angles, the angles at *E*

and *H* (E. Cor. 32. 1.) are, together, equal to two right angles: therefore (converse of E. 22. 3.) a circle may be described about the quadrilateral rectilineal figure ***EFHG***: and, consequently, (E. 21. 3.) the angle ***EFG*** is equal to ***EHG***: but (E. 8. 1. and construction,) the angle ***EFG*** is equal to ***BIL***; and (E. 21. 3.) ***BIL*** is equal to ***BDL***: wherefore, the angle ***EHG*** is equal to ***BDL***; and the angle ***BLD***, being (E. 31. 3.) a right angle, is equal to ***EGH***, which was made a right angle; also the side ***GE***, of the triangle ***EGH***, was made equal to the side ***BL***, of the triangle ***BLD***: wherefore (E. 26. 1.) ***EH*** is equal to ***BD***.

Prop. VIII.

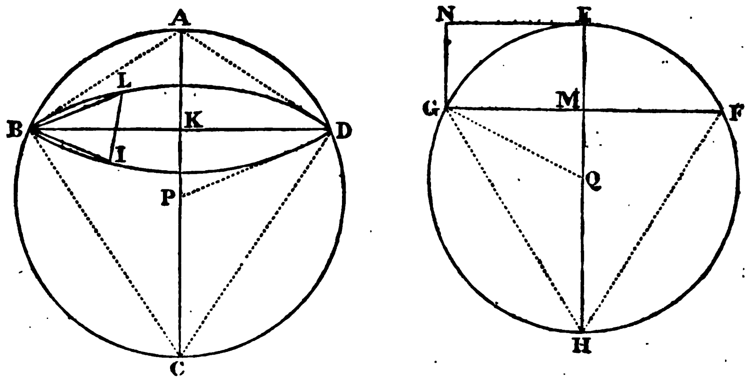
(62.) *Problem.* To draw a straight line, that shall be equal to a diameter of a given sphere.

Let *ABCD* * be the given sphere: it is required to draw a straight line, that shall be equal to a diameter of *ABCD*.

Take any two points, *A* and *B*, on the sphere's surface: and (Art. 59. 60.) from *A*, as a pole, describe a circle ***BLD***, passing through ***B***. Let ***BD*** be the diameter, and *K* the center of the circle ***BLD***; let ***ABCD*** be a great circle, passing through *A* and through ***BD***: and (Art. 61.) draw the straight line ***GF***, equal to ***BD***: let ***B***, *A* and ***D***, *A* be supposed to be joined; and

* See the figure in Art. 61.

make the triangle *FEG*, having its sides, *FE* and *GE*, equal to *DA* or *AB*, (Art. 31.): also, draw (E. 12. 1.)



EM at right angles to *FG*, and (E. 11. 1.) *FH* at right angles to *EF*, and let *EM* and *FH* meet in *H*. Then is *EH* equal to *AC*.

For, let *A*, *K* be supposed to be joined; let *AK*, produced, meet the sphere's surface in *C*, and let *D*, *C* be supposed to be joined: and since (Art. 22.) *AKC* is the axis of the circle *BLD*, it is a diameter of the sphere, and, therefore, *ADC* is a semi-circle; so that (E. 31. 3.) the angle *ADC* is a right angle, and therefore equal to the angle *EFH*, which was made a right angle.

Again, (E. 8. 1. and construction) the angle *ABD* is equal to *EFG*: wherefore (E. 21. 3. and 8. 6.) the angle *ACD* is equal to *EHF*; and *EF* was made equal to *AD*: therefore (E. 26. 1.) *EH* is equal to *AC*.

Prop. IX.

(63.) *Problem*. A circle being given, in a given

sphere, to find the direct distance between any point in its circumference, and either of its poles.

If the given circle be a great circle, find (Art. 62.) a straight line equal to the sphere's diameter; and upon it, as a diameter, describe a circle, in which (E. 4. 4.) inscribe a square: then (Art. 36.) the direct distance required is equal to the side of that inscribed square.

But, if the given circle be a lesser circle, let it be *BLD* *, and let *D* be any point in its circumference: let *K* be supposed to be its center, *A* one of its poles, *AKC* its axis; therefore (Art. 22.) *AC* passes through the center *P*, of the sphere, and *C* is the other pole of *ALD*: draw (Art. 62.) *EH* equal to *AC*: also draw *EN* (E. 11. 1.) at right angles to *EH*, and make (Art. 61.) *EN* equal to *KD*: upon *EH*, as a diameter, describe the circle *EGH*: through *N*, draw *NG* (E. 31. 1.) parallel to *EH*, and let it meet the circumference of *EGH* in *G*: lastly, join *A*, *B* and *B*, *C* and *H*, *G* and *E*, *G*. Then is *EG* equal to *AB*, and *HG* to *CB*.

For, draw (E. 12. 1.) *GM* parallel to *NE*: therefore, (E. 34. 1.) *MG* is equal to *EN*, and therefore it is equal also to *KD*: let *Q* be the center of the circle *EFGH*: and, join *P*, *D* and *Q*, *G*: and since, by the construction, *QG* is equal to *PD*, and *KD* to *MG*, and that the angles *PKD*, *QMG* are right angles, *QM* (E. 47. 1.) is equal to *PK*, and the angle *MQG* (E. 8. 1.) is equal to *KPD*: wherefore (E. 26. and 29. 3.) *EG* is equal to *AD*, and *HG* to *CD*.

* See the figure in Art. 62.

Prop. X.

(64.) ***Problem.*** An arch of a circle, in the surface of a given sphere, being given, to find the poles of the circle.

Find (Art. 63.) two straight lines equal to the distances between any point in the given arch and its two poles: then, from the extremities of the given arch, as poles, at distances equal to the two straight lines, so found, describe (Art. 59.) two circles in the sphere: the intersections of the circumferences of the circles, so described, are the poles of the circle, to which the given arch belongs.

For, the poles (Art. 31.) are in each of the circumferences, and therefore they are in the intersections of the circumferences.

(65.) Cor. A given arch of any circle, in a sphere, may be produced, in the sphere's surface (Art. 64. and 59.)

Prop. XI.

(66.) ***Problem.*** Two points in a sphere's surface, being given, to describe a circle, in the sphere, which shall pass through them, and shall have the direct distance between its pole and its circumference, equal to a given straight line *.

* In the following articles, when two points on a sphere's surface are directed to be joined, it is intended that they should be joined by arches of *great circles*, unless the contrary be specified.

From each of the two given points, as a pole, and at a distance equal to the given straight line, describe (Art. 59.) a circle in the sphere: then, from either of the two points, in which the circumferences of the circles, so described, cut one another, as a pole, describe (Art. 59.) a circle passing through either of the two given points; and it will, manifestly, pass through the other given point, and have the distance between its pole and its circumference equal to the given straight line.

(67.) Cor. 1. If the given straight line be either greater than a diameter of the given sphere, or less than the half of the straight line which joins the two given points in the sphere's surface, the construction fails; and, in that case, the problem is evidently impossible.

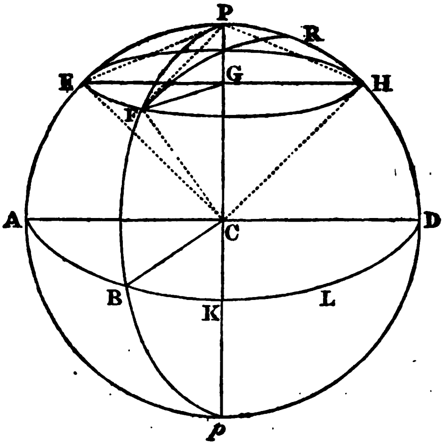
(68.) Cor. 2. An indefinite number of arches of great circles may, by means of Art. 63, be drawn through any given point, in a sphere's surface.

Prop. XII.

(69.) *Problem.* A spherical angle being given, in the surface of a given sphere, to find an arch of a great circle, in the sphere, which shall measure it.

Let *EPF* be a spherical angle, in the surface of the sphere *PApD*: It is required to find an arch of a great circle, which shall measure the angle *EPF*.

Find (Art. 63.), the direct distance between a great circle of the sphere and either of its poles; which is done,



independently of any great circle having been actually described: then, from *P* as a pole, at the distance so found, describe (Art. 59.) the great circle *ABD*; and (Art. 65.) produce the two arches *PE* and *PF*, until they meet the circumference *ABD*, in the points *A* and *B*.

The arch *AB* (Art. 54.) measures the spherical angle *EPF*.

PROP. XIII.

(70.) *Problem.* In the surface of a given sphere, to draw an arch of a great circle, which shall pass through a given point, in that surface, and be at right angles to the circumference of a given circle, in the sphere.

Find (Art. 64.) either pole of the given circle

describe (Art. 66.) a great circle, of the sphere, passing through the pole thus found, and through the given point: and (Art. 50.) its circumference shall be at right angles to the circumference of the given circle.

Prop. XIV.

(71.) *Problem.* A spherical angle, in the surface of a given sphere, being given, to make a plane rectilineal angle, which shall be equal to it.

From the angular point of the given spherical angle, as a pole, describe (Art. 59.) any circle, in the sphere; describe, also, (Art. 61.) a circle, in a plane, that shall be equal to the circle first described, in the sphere: and in this latter circle, place (E. 1. 4.) a straight line equal to the direct distance between the two points, in which the circle in the sphere, first described, cuts the arches containing the given angle; which distance (Art. 59.) may be considered as given: then shall the plane rectilineal angle subtended by this straight line, at the center of the circle in which it is placed, be equal (E. 28. and 27. 3. and Art. 54.) to the given spherical angle.

PART I.

THE ELEMENTS OF

Spherical Geometry.

SECTION III.

ON THE GENERAL RELATIONS OF THE SIDES AND ANGLES OF SPHERICAL FIGURES.

DEFINITION.

(72.) A *Spherical Triangle* is a figure, on the surface of a sphere, contained by three arches of great circles, in the sphere, each of which arches is less than the semi-circumference of a great circle *.

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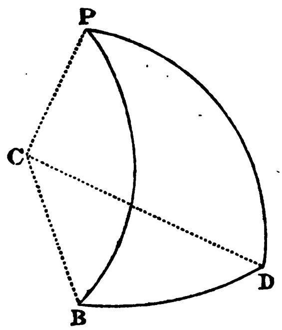
* A portion of the sphere's surface may be bounded by three arches of great circles, of which arches, one may be greater than the half of the circumference; and the angle opposite to it may be greater than two right angles. There might, indeed, if the restriction laid down in the above definition were removed, be no fewer than eight spherical triangles formed, by joining three given points on a sphere's surface. But, as the main object of Spherical Geometry is to elucidate Spherical Trigonometry, and as the determination of the unknown parts, of such a trilateral figure, is always reducible to the solution of a spherical triangle, such as we have defined it to be, the properties of the former kind of figure are not investigated in this Treatise.

(73.) Cor. The three sides, therefore, of a spherical triangle are together less than six quadrants: and, (Art. 42), each of its angles being less than two right angles, its three angles are together less than six right angles.

Prop. I.

(74.) *Theorem.* The three sides of a spherical triangle are together less than the circumference of a great circle of the sphere: but any two of them are greater than the third.

Let *PBD* be a spherical triangle: the three sides



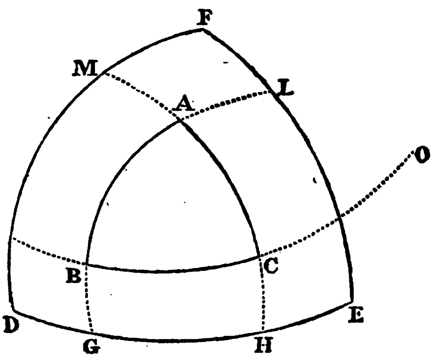
PD, *BD*, and *DP* are, together, less than the circumference of a great circle, but any two of them are greater than the third.

For let *C* be the sphere's center; and let *C*, *B*, *C*, *D* and *C*, *P* be supposed to be joined: Then, it is evident (Art. 7.) that *CB*, *CD* and *CP* will be the intersections of the planes of the three great circles, the arches of which bound the spherical triangle *PBD*: it is evident, also,

(E. 33. 6.) that the sides *PB, BD, DP* are the measures of the plane angles *PCB, BCD,* and *DCP*, respectively, by which angles the solid angle at *C* is contained: but (E. 21. 11.) the three plane angles are together less than four right angles; and (E. 20. 11.) any two of them are greater than the third: wherefore, also, any two sides of the spherical triangle are greater than the third, and the three sides, together, are less than the measure of four right angles, that is, they are less than the circumference of a great circle of the sphere.

(75.) COR. In the same manner, it may be shewn, that the aggregate of the sides of a spherical polygon, is less than the circumference of a great circle of the sphere, if the polygon be bounded by arches of great circles, each of which is less than the semi-circumference of a great circle.

(76.) DEF. If, about each of the angular points of a spherical triangle, as a pole, a great circle, in the sphere, be described, the spherical triangle included by the circumferences of the circles, so described, is called the *Polar Triangle* of the given spherical triangle.



Thus, if *ABC* be a spherical triangle, and *DE*, *EF*, and *FD*, be the arches of great circles, described from *A* *B* and *C* as their respective poles, the spherical triangle *DEF* is called the Polar Triangle of *ABC* *.

(77.) Cor. 1. The angular points of the polar triangle are also (Art. 27.) the poles of the great circles, the arches of which include the given triangle; and, therefore, if any sides of the given triangle, as *AB*, be produced to meet the sides of the polar triangle, in *G* and *L*, the side so produced, *GL*, will measure (Art. 54.) the opposite angle *E*, of the polar triangle; and if two sides, as *AB* and *AC*, of the given triangle be produced, the segment *GH* which they cut off from the side *DE* of the polar triangle, will likewise measure the angle *A*, of the given triangle, that is opposite to *DE*.

It is also manifest, from Art. 50. that an arch of a great circle, drawn from any angle of either triangle, as from the angle *A*, of the triangle *ABC*, at right angles to the opposite side *BC*, of that triangle, will likewise cut the opposite side *DE*, of the other triangle, perpendicularly, and will pass through the opposite angle *F* of the other triangle: and conversely.

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* For the sake of simplicity in the figure, each of the sides of the triangle *ABC* is here supposed to be less than a quadrant, and each of its angles to be less than a right angle: in which case, the triangle, *ABC* lies wholly within its polar triangle. But if any of the sides of the triangle *ABC* be greater than a quadrant, the sides of the two triangles will intersect one another. The same reasoning, however, which is applied in the one case, is equally applicable in the other.

(78.) COR. 2. In either of the two triangles, the measure, *GH*, of any angle, *A*, of the one, together with the side, *DE*, of the other, that is opposite to that angle, is equal to the semi-circumference of a great circle of the sphere.

For (Art. 36. and 7.) *DH* * and *GE* are quadrants, and are, therefore, (Art. 35.) together equal to the semi-circumference of a great circle: but *DH* together with *GE* is manifestly equal to *DG*, and *HE*, and twice *GH*, taken together; that is, to *DE* and *GH* together: therefore, *DE* and *GH* are, together equal to the semi-circumference of a great circle. And, in the same manner, may the measure of any angle of the triangle *DEF*, together with the side of *ABC*, opposite to that angle, be shewn to be equal to the semi-circumference of a great circle of the sphere.

Hence, the aggregate of any two sides, in either triangle, together with the aggregate of the measures of the two angles of the other triangle, that are opposite to them, is equal to the circumference of a great circle: also, the difference of any two sides, of the one triangle, is equal to the difference of the measures of the angles, of the other triangle, that are opposite to them.

And, if an arch of a great circle be drawn from any angle of either triangle, as from the angle *A* of *ABC*, at right angles to the opposite side *BC* of that triangle, the difference of the segments, into which it divides the

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* See the figure in Art. 76.

opposite side DE, of the other triangle, is equal to the difference of the measures of the segments, into which it divides the angle A.

(79.) Cor. 3. If a spherical polygon be given, which is bounded by arches of great circles, each less than a semi-circumference, and about each of its angular points, as a pole, a great circle be described, another spherical polygon will thus be formed: and it may be shewn, in the same manner as before, that in either of the two figures, the measure of any angle, in the one, together with the side opposite to it of the other, is equal to the semi-circumference of a great circle of the sphere.

Prop. II.

(80.) *Theorem.* The three angles of any given spherical triangle are together greater than two right angles.

For, (Art. 78.) the measures of the three angles of the given triangle, together with the three sides of its polar triangle, are equal to three semi-circumferences of a great circle: but (Art. 74.) the three sides of the polar triangle are less than two such semi-circumferences: wherefore the measures of the angles of the given triangle are, together, greater than the remaining semi-circumference: and the angles themselves are, consequently, together, greater than two right angles.

(81.) Cor. 1. Hence, and from Art. 42, it is evident, that the exterior angle, of a spherical triangle, is

less than the two interior opposite angles: and, that the difference between the exterior angle, and the two interior opposite angles, is equal to the excess of the three angles of the triangle above two right angles.

(82.) Cor. 2. Hence, in a right-angled spherical triangle, the two angles at the hypotenuse are, together, greater than a right angle.

(83.) Cor. 3. If the three angles of a spherical triangle be equal to one another, the measure of any one of them exceeds the third part of the semi-circumference of a great circle of the sphere.

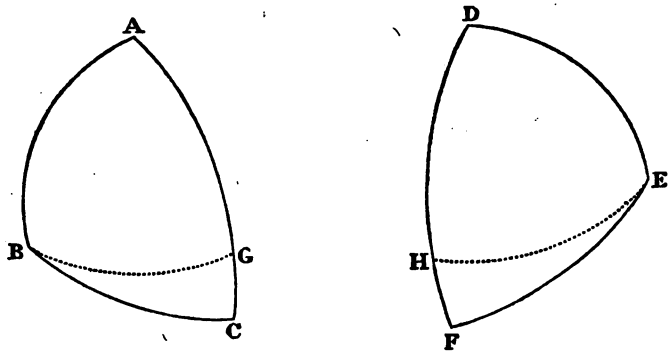
(84.) Cor. 4. If, therefore, a spherical triangle be equilateral, none of its sides (Art. 78.) can exceed two-thirds of the semi-circumference, or one-third of the circumference of a great circle.

(85.) Cor. 5. By a method, similar to that used in demonstrating the first corollary of the thirty-second proposition of Euclid's Elements, and by the help of Art. 80. it may be shewn, that all the interior angles of any given spherical polygon, together with four right angles, are greater than twice as many right angles as the figure has sides; the given polygon being bounded by arches of great circles, each of which is less than a semi-circumference.

PROP. III.

(86.) *Theorem.* If two spherical triangles * on the same sphere, or on equal spheres, have the three sides of the one equal to the three sides of the other, each to each, the angles also of the one shall be equal to the angles of the other, each to each, to which the equal sides are opposite.

Let *ABC, DEF* be two spherical triangles, on the same sphere, or on equal spheres; and let the side *AB*



be equal to *DE*, and *BC* to *EF*, and *AC* to *DF*: then, the angle *ABC* is equal to *DEF*, *BCA* to *EFD*, and *CAB* to *FDE*.

For, from *A* and *D* as poles, at the equal distances *AB* and *DE*, describe (Art. 59.) the circles *BG, HE*, which (Art. 38.) are therefore equal circles:

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* In the subsequent propositions, when two spherical triangles are compared, it is supposed that they belong to the same sphere, or to equal spheres, unless the contrary be specified.

If AB be equal to AC, and DE to DF, the circles BG and EH will pass through the points C and F respectively: and because, by the hypothesis, BC is equal to EF, the arches of the circles, described from the poles A and D, at the distances AB, and DE, intercepted between B, C, and E, F, will (Art. 17.) be equal to one another: and therefore, (Art. 54. 55.) the spherical angle BAC is equal to EDF. But, if AB be not equal to AC, let AC be the greater; consequently, DF is greater than DE: let, therefore, the circle BG meet AC in G, and let the circle EH meet DF in H: therefore (Art. 38.) AG is equal to DH: and the whole AC is supposed to be equal to the whole DF; therefore the remainder GC is equal to the remainder HF: also, the side BC is supposed to be equal to EF: and BG, EH are arches of equal circles; therefore, (Art. 40.) BG is equal to EH; and consequently, (Art. 54. 55.) the spherical angle BAC is equal to EDF.

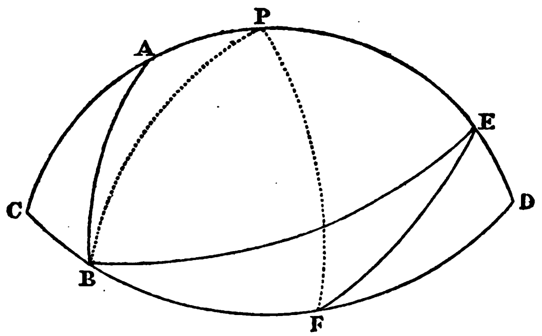
In the same manner also, may the other angles of the triangle BAC be shewn to be equal to the other angles of the triangle EDF, each to each, to which the equal sides are opposite.

(87.) Cor. If the arch of a great circle cut the semi-circumferences of two great circles, in a sphere, so as to make their alternate segments equal, it shall also make the alternate spherical angles equal:

For, first, let DEC, DBC, two semi-circumferences of great circles, be cut by the arch EB of a great circle, so that the segment CE, is equal to DB, and consequently,

the segment *ED* equal to *BC*: then, the spherical angle *CEB* is equal to the alternate angle *EBD*.

For, *EB* is common to the two triangles *CBE*,



DEB; and the sides *CB*, *CE* are equal to the sides *DE*, *DB*, each to each: wherefore (Art. 86.) the angle *CEB* is equal to *EBD*, and the angle *CBE* to *BED*.

Prop. IV.

(88.) *Problem.* Upon a given arch of a great circle, on the surface of a given sphere, to describe an equilateral spherical triangle: the given arch not being greater than a third part of the circumference of a great circle *.

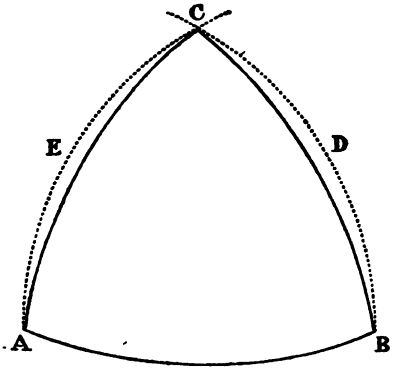
Let *AB* † be the given arch of a great circle, not greater than a third part of the circumference: it is required to describe an equilateral triangle on *AB*.

From *A*, as a pole, at the distance *AB*, describe (Art. 59.) the circle *BDC*; and from *B* as a pole, at the

* It appears, from Art. 84. that this is a necessary restriction.

† See the figure in the next page.

distance *BA*, describe the circle *AEC*: then, if *AB* be a quadrant, *BDC* and *AEC* are (Art. 36.) arches of great circles: but if *AB* be not a quadrant, join (Art. 66.)



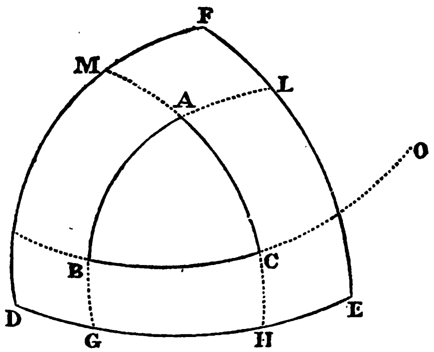
A, *C* and *B*, *C* by the arches of great circles *AC* and *BC*. The triangle *ACB* is equilateral. For (Art. 31. 17. and construction), *AC* and *BC* are each of them equal to *AB*; and consequently they are equal to one another.

(89.) Cor. 1. By the same method, upon any given arch of a great circle, as a base, an isosceles spherical triangle may be described, having each of its equal sides less than the base, but greater than the half of the base.

(90.) Cor. 2. By means, also, of Art. 88. and 63, an equiangular spherical triangle may be described, on the surface of a given sphere, having each of its angles equal to a given spherical angle, which is greater than a third part of two right angles.

For find (Art. 69.) an arch *OC* of a great circle

which measures the given spherical angle: and produce



(Art. 65.) *OC* to *B*: describe (Art. 68.) any great circle passing through the point *O*, and let it cut *OB* in *B*: wherefore (Art. 7.) *OCB* is the semi-circumference of a great circle: upon *BC* describe the equilateral triangle *ABC*: lastly, (Art. 63.) describe *DEF* the polar triangle of *ABC*. *DEF* is a spherical triangle, having each of its angles, *D*, *E*, and *F* equal to the given angle.

For (Art. 78.) the measure of the angle *F*, together with *BC*, is equal to the semi-circumference *BCO*; take from both the common part *BC*, and the measure of the angle *F* is equal to *CO*, which was made the measure of the given spherical angle: wherefore, the angle *F* is equal to the given angle: and it is plain (Art. 78.) that the angles *D*, *E*, *F* are all equal, because the sides of the triangle *ABC* are, by the construction, all equal.

Prop. V.

(91.) *Problem.* From a given point, on the surface of a given sphere, to draw an arch of a great circle, equal to a given arch of a great circle.

The problem is solved, by means of Art. 88, and some of the preceding articles, exactly in the same manner, as the second proposition of the first Book of Euclid's Elements.

Prop. VI.

(92.) *Problem.* From the greater of two given arches of great circles, in a sphere's surface, to cut off a part equal to the less.

The method of solution is the same as that of E. 3. 1. and is dependent upon Art. 91. 59. and 31.

(93.) Cor. Hence, and by means of Art. 89. an arch of a great circle may be drawn from a given point, in a sphere's surface, which shall be equal to the aggregate of two such given arches, or to the excess of the greater of them above the less.

Prop. VII.

(94.) *Problem.* On the surface of a given sphere, to describe a spherical triangle, of which the sides shall be equal to three given arches of great circles: but any two whatever of these must be greater than the third*.

It is evident, that Art. 92. 59. and 31. may be applied to the construction of this problem, and its demonstration, exactly in the same manner as the corresponding propositions are employed in E. 22. 1.

* This restriction is indicated by Art. 74.

(95.) SCHOLIUM. If one of the given arches be taken as the base of the triangle to be described, it is evident, that either of the other sides may be made to terminate in the right, or in the left, extremity of the base: and thus, on the same side of that base, if the two sides be unequal, there may be made two spherical triangles, equally answering the conditions of the problem, but having their sides not in the same order.

PROP. VIII.

(96.) *Problem.* At a given point in a given arch of a great circle, to make, in the surface of a given sphere, a spherical angle equal to a given spherical angle.

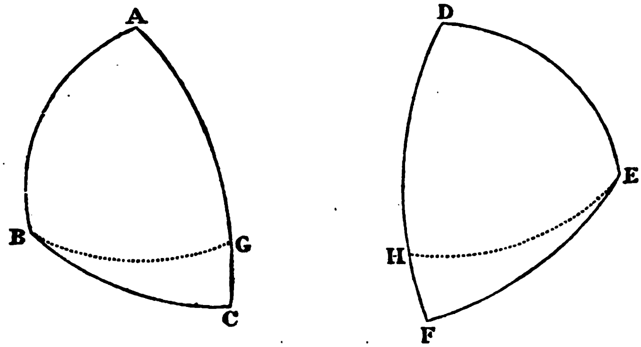
If, by means of Art. 59. and 94. a construction be made, exactly similar to that made in E. 23. 1. it will follow immediately from Art. 86. that a spherical angle has been made, in the sphere's surface, equal to the given angle.

PROP. IX.

(97.) *Theorem.* If two spherical triangles have two sides of the one equal to two sides of the other, each to each, and have, also, the included angles equal, their third sides shall be equal, and their remaining angles, also, shall be equal, each to each, to which the equal sides are opposite.

Let *ABC, DEF* be two spherical triangles, on equal spheres, or on the same sphere, which have the sides

AB, *AC* equal to *DE*, *DF*, each to each, and the angle



BAC equal to the angle *EDF*; the base, *BC*, shall be equal to the base, *EF*; and the other angles, to which the equal sides are opposite, shall be equal, each to each.

From *A* and *D*, as poles, at the equal distances *AB*, *DE* describe (Art. 59.) the circles *BG* and *HE*; which (Art. 38.) are, therefore, equal circles.

If *AB*, be equal to *AC*, and *DE* to *DF*, the circles *BG* and *EH* will pass through the points *C* and *F* respectively: and because the angle *BAC* is equal to *EDF*, the arch *BG* will (Art. 55.) be equal to *EH*, and therefore (Art. 17.) the side *BC*, will, in that case, be equal to *EF*.

But, if *AB* be not equal to *AC*, let *AC* be greater than *AB*; consequently, *DF* is greater than *DE*: let, therefore, the circle *BG* meet *AC* in *G*; and let the circle *EH* meet *DF* in *H*: and since, by the hypothesis, *AC* is equal to *DF*, and *AG* (Art. 38.) to *DH*, therefore, *GC* is equal to *HF*: and *BG* and *EH*, have been

shewn to be equal arches of equal circles: wherefore, BC is equal (Art. 40.) to EF; and therefore, (Art. 86.) the angle B is equal to E, and the angle C to F.

Prop. X.

(98.) *Theorem.* If two spherical triangles, have two angles of the one equal to two angles of the other, each to each, and have a side of the one equal to a side of the other, namely, the sides adjacent to the equal angles, the other sides of the two triangles shall be equal, each to each, and the third angle of the one, equal to the third angle of the other.

The proposition may be deduced from Art. 97. by means of Art. 78: or it may be demonstrated, *ex absurdo*, as is the twenty-sixth proposition, of the first Book of Euclid's Elements. (Art. 99.)

Prop. XI.

(99.) *Theorem.* If two spherical triangles have the three angles of the one equal to the three angles of the other, each to each, the three sides of the one shall, also, be equal to the three sides of the other, each to each, to which the equal angles are opposite.

For, it is manifest, from Art. 78. that the sides of the polar triangles, of the two given triangles, will be equal, each to each; consequently, (Art. 86.) the angles, also, of the polar triangles are equal, each to each: and, therefore, (Art. 78.) the sides of the given triangle are equal, each to each.

Or, the proposition may be proved, independently of

the polar triangle (by the help of Art. 97. and 98.), by producing two sides of either of the given triangles, until the parts produced, are equal to the two sides about the equal angle of the other triangle, each to each; and by producing the third side, also, of the first mentioned triangle, until it meet the circumference of the great circle, which joins the parts produced.

(100.) Cor. If the arch of a great circle cut the semi-circumferences of two great circles, in a sphere, so as to make the alternate spherical angles equal, the alternate segments of the semi-circumferences shall, also, be equal. (Art. 99. 56.)

Prop. XII.

(101.) *Problem.* To bisect a given angle, in a sphere's surface, contained by the arches of any two equal circles.

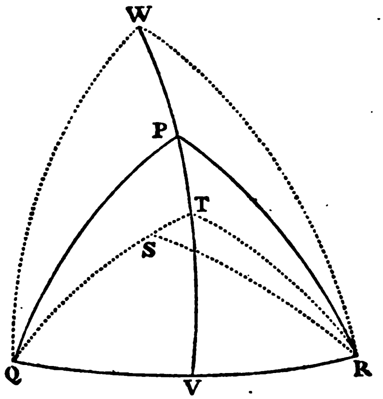
The problem is solved by the same construction as is the ninth proposition of the first Book of Euclid's Elements: and if the arches which contain the given angle, be arches of great circles, the demonstration rests solely on Art. 86: but if the given angle be contained by arches of lesser circles, it is necessary, first to join, by arches of great circles, the angular point, and the extremities of the parts cut off, in the construction, from those arches: and the proposition may then be demonstrated by means of Art. 86. and 17.

Prop. XIII.

(102.) *Problem.* To bisect a given circular arch, in the surface of a given sphere.

Let *QR* be the given arch of any circle, in a sphere: it is required to bisect it.

Find (Art. 64.) either pole, *P*, of *QR*, and join



(Art. 66.) *P*, *Q* and *P*, *R*: bisect (Art. 101.) the spherical angle *QPR*, by the arch of a great circle *PV*: *QR* is bisected in *V**.

For, since *P* is the pole of the circle *QR*, *QV* and *VR* are (Art. 54.) the measures of the angles *QPV*, *RPV*; and these angles are, by the construction, equal to one another: wherefore, their measures *QV* and *VR* are equal; that is, *QR* is bisected in *V*.

* If the given arch exceed the half of the circumference of its circle, the operation may be applied to the remainder of the circumference.

(103.) Cor. 1. Any point *T*, or *W*, in the circumference *VP*, of a great circle, which bisects at right angles the arch *QR*, of any circle in a sphere, is equidistant from the two extremities of that arch.

For, join *Q*, *T* and *R*, *T*: let *P* be the pole of *QR*, which (Art. 50.) is in *VP*; and join *P*, *Q* and *P*, *R*:

Then, it is manifest, from Art. 102. and 16. that the arch *PV* bisects the spherical angle *QPR*: and, since *PT* is common to the two triangles *PTQ*, *PTR*, and that, *PQ* (Art. 31.) is equal to *PR*, and the angle *QPT* equal to the angle *RPT*, therefore (Art. 97.) *TQ* is equal to *TR*.

(104.) Cor. 2. Any point, in the sphere's surface, that is not in the circumference of a great circle, which bisects any given arch at right angles, is not equidistant from both the extremities of that arch.

For, if it be possible, let *S* be equidistant from *Q* and *R*; join *Q*, *S* and *R*, *S*: and let *QS* meet the arch *VP*, which bisects *QR* at right angles, in *T*: join *T*, *R*.

Then, by the hypothesis, *QS* is equal to *SR*; therefore *QS* and *ST* are, together, equal to *SR* and *ST*, that is, *RS* and *ST* are together equal to *QT*, that is, (Art. 103.) to *TR*: but (Art. 74.) *RS* and *ST* are greater than *TR*: which is absurd.

Prop. XIV.

(105.) *Theorem.* If two angles of a spherical triangle be equal to one another, the sides, also, which subtend the equal angles, shall be equal to one another.

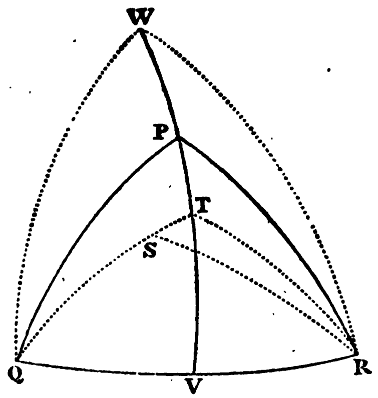
The proposition is proved, *ex absurdo*, by means of Art. 97. exactly in the same manner, as the sixth proposition of the first Book of Euclid's Elements.

(106.) Cor. Every equiangular spherical triangle is also equilateral.

Prop. XV.

(107.) *Theorem.* The angles at the base of an isosceles spherical triangle are equal to one another.

Let *TQR* be an isosceles spherical triangle, of



which the side *TQ* is equal to *TR*: the spherical angle *TQR* is equal to *TRQ*.

For, draw (Art. 101.) the arch *TV* bisecting the angle *QTR*: then, since *QT*, *TV* are equal to *RT*, *TV*, each to each, and the angle *QTV* to *RTV*, the other angles of the triangle *QTV* are equal to the other angles of *RTV* (Art. 97.); that is, the angle *TQR* is equal to *TRQ*.

Or the proposition may be deduced from Art. 78. and 105, by describing the polar triangle of any given isosceles spherical triangle.

Or, the proof of the proposition very readily follows, from Art. 40, if arches of circles be first described from each extremity of the base, as a pole, at a distance equal to either of the equal sides of the isosceles spherical triangle.

(108.) Cor. 1. Every equilateral spherical triangle is also equiangular.

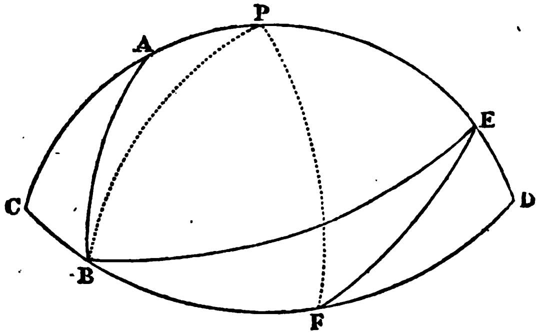
(109.) Cor. 2. It has appeared, from the first mode of demonstrating Art. 107, that the arch of a great circle, which bisects the vertical angle of an isosceles spherical triangle, bisects its base at right angles: it is plain, therefore, (Art. 18. 49.) that the arch of a great circle, which joins the vertex and the bisection of the base, cuts the base perpendicularly.

Prop. XVI.

(110.) *Theorem.* One side of a spherical triangle having been produced, if the exterior angle be equal to the interior opposite angle, the two other sides, together, shall be equal to the semi-circumference of a great circle: and conversely, if the two other sides be, together, equal to such a semi-circumference, then the exterior angle shall be equal to the interior opposite angle.

Let *PCB* be a spherical triangle, and let any one of

its sides, *CB*, be produced: if the angle *PBD* be equal



to the angle *C*, *CP*, together with *PB*, is equal to a semi-circumference: and, if *CP*, together with *PB*, be equal to a semi-circumference, then the angle *PBD* is equal to *C*.

First, let the angle *PBD* be equal to *C*: and let *CP* produced meet *CBD* in *D*: then (Art. 56.) the angle *D* is equal to *C*, and therefore, also, equal to *PBD*: consequently, (Art. 105.) *PB* is equal to *PD*; but (Art. 7.) *PD* and *PC* are, together, equal to a semi-circumference: therefore, also, *PB* and *PC* are, together, equal to the same semi-circumference, of a great circle.

Secondly, let *CP* and *PB* be, together, equal to a semi-circumference, that is, (Art. 7.) to *CPD*; take away from both *CP*, and there remains *PB* equal to *PD*: wherefore, (Art. 107.) the angle *PBD* is equal to the angle *D*, that is, (Art. 56.) to *C*: and *PBD* is the exterior, and *C* the interior opposite angle, of the spherical triangle *PBC*.

Prop. XVII.

(111.) *Theorem.* If two spherical triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by those two sides of the one, greater than the angle contained by the two sides equal to them of the other, the base of that which has the greater angle shall be greater than the base of the other.

The proposition is proved, by the help of the preceding articles, in the same manner as is the twenty-fourth proposition of the first book of Euclid's Elements.

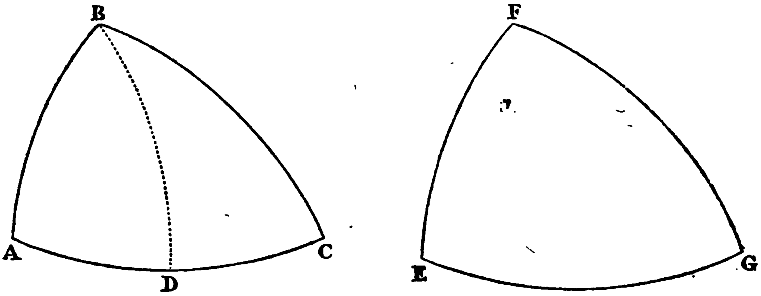
(112.) Cor. From the proposition itself may be deduced its converse; in the same manner as the twenty-fifth proposition of the first book of Euclid's Elements is deduced from the twenty-fourth of that book.

Prop. XVIII.

(113.) *Theorem.* If two spherical triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, the sides opposite to equal angles in each, and if the sides subtending the other two equal angles be, together, unequal to two quadrants, then, the remaining sides shall be equal, each to each; and the third angle of the one, to the third angle of the other.

Let *ABC*, *EFG* be two spherical triangles, having

the angle *A* equal to *E*, the angle *C* equal to *G*, and the sides *BC*, *FG*, which subtend equal angles, also equal:



and let the two sides *BA* and *FE*, which subtend the other equal angles, be, together, either greater, or less, than two quadrants: then shall *BA* be equal to *FE*, *AC* to *EG* and the angle *ABC* to the angle *EFG*.

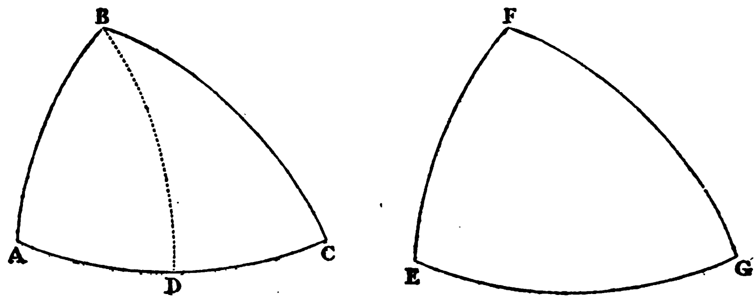
For, if *AC* be unequal to *EG*, let *AC* be the greater; from *AC* cut off *CD* equal to *EG*, and join (Art. 66.) *B*, *D*; then, since the two sides *BC*, *CD* are severally equal to *FG*, *GE*, and the angle *BCD* to *FGE*, therefore (Art. 97.) the angle *BDC* is equal to *FEG*, and *BD* to *FE*; but the angle *FEG*, by the supposition, is equal to *BAC*; wherefore, the angle *BDC* is equal to *BAC*; and consequently, (Art. 100.) *AB* and *BD* are, together, equal to a semi-circumference; but *BD* has been proved to be equal to *FE*: wherefore, *BA* and *FE* are, together, equal to a semi-circumference: which is contrary to the hypothesis. So that *AC* and *EG* are not unequal; therefore, (Art. 97.) *BA* is equal to *FE*, and the angle *ABC* to *EFG*.

Prop. XIX.

(114.) *Theorem.* If two spherical triangles have one angle of the one equal to one angle of the other, and have also the two sides about another angle in each severally equal, and if the third angle, in each, be either greater or less than a right angle, the remaining side of the one triangle shall, also, be equal to the remaining side of the other, and the other angles to the other angles, each to each.

This may be inferred from Art. 113. by the help of polar triangles: or it may be proved, *ex absurdo*, the same construction having first been made, as in Art. 113.

For, let the angle C in the triangle ABC, be equal to G, in EFG; also let AB, BC be equal to EF, FG,



each to each; and let the angles A and E not be right angles: then if AC be not equal to EG, make CD equal to it, and join B, D. Then (Art. 97.) BD is equal to FE, and the angle BDC equal to E: but AB is supposed to be equal to FE; wherefore, AB is equal to BD; and therefore, (Art. 107.) the angle A is equal

to BDA; again, (Art. 42.) the angles BDC, BDA are equal to two right angles; and the angle E has been shewn to be equal to BDC, and the angle A to BDA: therefore, A and E are together equal to two right angles, which is impossible: for, by the hypothesis, each of them is either greater or less than a right angle. Therefore, AC is equal to EG, and (Art. 86.) the other angles, of ABC, are equal to the remaining angles, of EFG; each to each.

Prop. XX.

(115.) *Theorem.* If two spherical triangles have two angles of the one equal to two angles of the other, each to each, and the two sides about the third angle of the one, not quadrants, but equal to the two sides about the third angle of the other, each to each, then shall the remaining side be also equal to the remaining side, and the remaining angle to the remaining angle.

In the two spherical triangles ABC*, EFG, let the angles A, C be equal to the angles E, G, each to each; and let the sides AB, BC, which are not quadrants, be equal to EF, FG, each to each: the remaining sides AC, EG are equal, and the remaining angles ABC and EFG are also equal.

For, if AC and EG be unequal, let either of them, as AC, be the greater, and from AC cut off (Art. 92.) CD equal to GE, and join (Art. 66.) B, D.

Then, since BC, CD are equal to FG, GE, each to each, and the angle BCD to the angle FGE, there-

* See the figure in Art. 114.

fore, (Art. 97.) *BD* is equal to *FE*, and the angle *BDC* to *FEG*: But, by the supposition, *FE* is equal to *BA*, and the angle *FEG* to *BAC*: wherefore, *BA* is equal to *BD*, and the angle *BDC* to *BAC*: also, because *BA* is equal to *BD*, the angle *BAD*, or *BAC*, is equal (Art. 107.) to *BDA*; wherefore, *BDA* is equal to *BDC*, and (Art. 42.) the angle *D*, and consequently, also, the angle *A*, is a right angle: Therefore, (Art. 51.) *B* is the pole of the circle *AC*; and (Art. 36.) *BA* is a quadrant: which is contrary to the hypothesis. Therefore, *EG* is not unequal to *AC*; and (Art. 86.) the angles *ABC* and *EFG* are equal to one another.

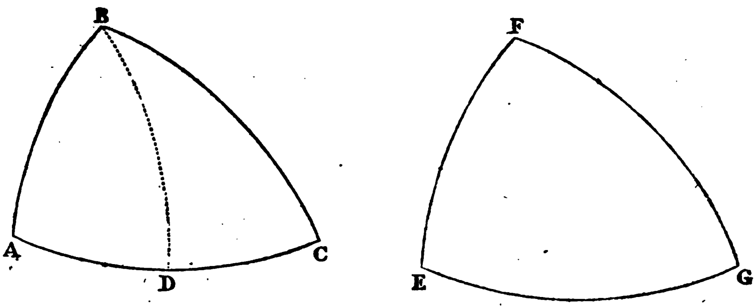
PROP. XXI.

(116.) *Theorem.* If in two right-angled spherical triangles, having only one angle in each a right angle, the hypotenuses be equal, and if another side, or another angle, not the right angle, in the one, be equal also to another side, or another angle, in the other, the remaining angles and sides shall be equal, in either case, each to each.

Let *BAC*, *FEG* be two spherical triangles, having only the angles *A* and *E* right angles, and the hypotenuse *BC* equal to the hypotenuse *FG*; and, first, let another side, as *BA*, of *BAC*, be equal to the side *FE*, of *FEG*: then, shall *AC* be equal to *EG*, and the other angles, of *ABC*, to the other angles, of *EFG*.

For, if *AC* be unequal to *EG*, let it be the greater: and make (Art. 92.) *AD* equal to *EG*, and join *B*, *D* (Art. 66.)

Then (Art. 97.) *BD* is equal to *FG*, and is, therefore, equal to *BC*, because the triangles are supposed to



have equal hypotenuses: therefore, *BDC* is an isosceles triangle; and (Art. 109.) the arch of a great circle, joining *B* and the bisection of *DC*, cuts *DC* perpendicularly, as, by the hypothesis, does *BA*; therefore (Art. 51.) *B* is the pole of *AC*, and (Art. 50.) the angle *BCA* is a right angle; which is contrary to the hypothesis. Since therefore, *AC* is not unequal to *EG*, the two right-angled triangles have all the sides of the one equal to the sides of the other, each to each; and (Art. 86.) their angles are equal, each to each.

Next, let one other angle, as *C*, of *ABC*, be equal to an angle, *G*, of *EFG*. Then, if the angles *ABC*, *EFG* be unequal, let *ABC* be the greater, and make (Art. 96.) the angle *CBD* equal to *EFG*: and since the angle *G* is equal to *C*, and *GFE* to *CBD*, and *FG* to *BC*, by the hypothesis, therefore, (Art. 98.) the angle *BDC* is equal to the right angle *FEG*; and *BAC* is a right angle; wherefore, (Art. 51.) *B* is the pole of

AC, and the angle *BCA* is a right angle, which is contrary to the supposition.

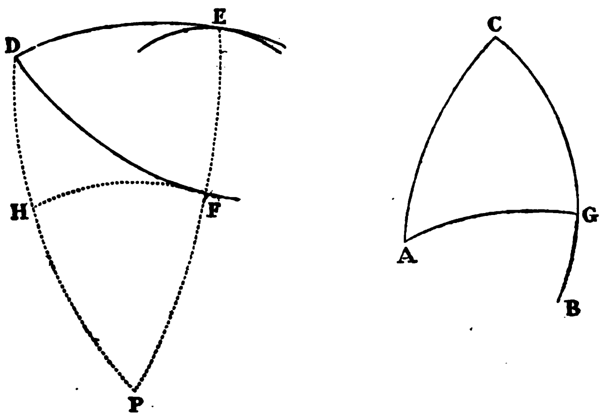
Since, therefore, the angle *ADC* is not unequal to *F*, the sides of the triangle *ABC* are (Art. 98.) equal to the sides of *EFG*.

Prop. XXII.

(117.) *Problem.* Through a given point, on the surface of a sphere, to draw an arch, of a great circle, which shall make, with the circumference of a given great circle, an angle equal to a given spherical angle.

Let *A* be the given point, *BC* the given circumference, and *EDF* the given spherical angle. It is required to draw, from *A*, an arch of a great circle, that shall make, with *BC*, an angle equal to *EDF*.

From *A*, draw (Art. 70.) the arch *AG* perpendicular



to *BC*; and, if the angle *EDF* be a right angle, the

angle AGC has been made equal to it: but, if EDF be not a right angle, find (Art. 64.) the pole P of DE; join P, D; from DP (Art. 92.) cut off DH, equal to AG; from P, as a pole, at the distance PH, describe (Art. 59.) the circle HF, cutting DF in F; join P, F (Art. 66.) and produce PF to meet DE in E: lastly, from A, as a pole, at the distance DF, describe a circle, cutting BC in C; and join A, C: the angle ACG is equal to EDF.

For, (Art. 50.) the angle E is a right angle: and (Art. 32.) the arch FE is equal to HD, which was made equal to AG; also, by the construction, AC is equal to DF: so that the right-angled triangles ACG, DEF have equal hypotenuses, and have, also, two other sides, namely, AG and FE equal: wherefore, (Art. 116.) the angle ACG is equal to the given spherical angle EDF.

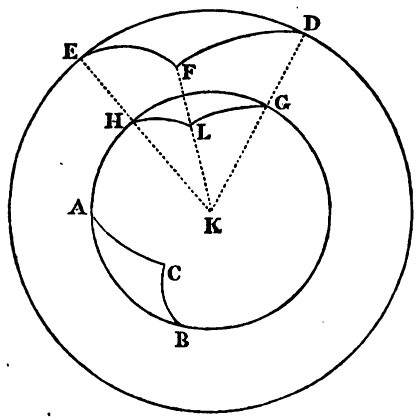
(118.) Scholium. If AG be a quadrant, that is, if A be the pole of BC, the construction necessarily fails: for, it is not possible, in that case, to draw from A an arch of a great circle which shall make with BC any other angle (Art. 50.) than a right angle. That case, therefore, is tacitly excluded. Again, the construction, as the figure is drawn, supposes DH to be the less of the two distances, of A from the circumference of BC: but, then (Art. 56.) AG will make equal angles with that circumference at both intersections.

Prop. XXIII.

(119.) If the sides of two spherical triangles, on unequal spheres, be similarly posited, and the three angles

of the one be equal to the three angles of the other, each to each, the sides about the equal angles are proportionals; and those, which are opposite to the equal angles, are homologous sides.

Let the spherical triangles *ABC*, *EDF*, be on unequal spheres; and let the angles *A*, *B*, *C* of the one, be equal to the angles *E*, *D*, *F* of the other, each to each; namely, *A* to *E*, *B* to *D*, and *C* to *F*. The sides about the equal angles of the triangles *ABC*, *DEF* are proportionals. For, let the spheres be supposed to be placed, so as to



have their centers in the same point *K*: Suppose, also, *EK*, *DK* and *FK*, to be the common sections of the planes of the great circles *ED*, *DF* and *FE*; and *HG*, *GL* and *LH* to be the common sections of those planes, and the surface of the lesser sphere *ABG*.

Then (Art. 53.) the angles of the spherical triangle *GHL* are equal to the angles of *DEF*, and are, therefore, by the hypothesis, equal to the angles of *ABC*, each to

each: wherefore (Art. 99.) *GH* is equal to *BA*, *HL* to *AC*, and *LG* to *CB*.

Again, it follows from E. 33. 6. that *GH* has to the whole circumference, of which it is a part, the same ratio, as *DE* has to the whole circumference, of which *DE* is a part: therefore, (E. 16. 5.) *GH* and *DE* are to each other, as the circumferences of great circles, in their respective spheres: and the same ratio, it may be, likewise, shewn, has *HL* to *EF*, and *LG* to *FD*: therefore, (E. 11. 8.) the sides of the two triangles *GHL*, *DEF*, about equal angles, are proportionals, and those are homologous sides, which are opposite to the equal angles. But the sides and the angles of the triangle *ABC* have been proved to be equal to the sides and the angles of *GHL*: wherefore, the sides of the triangles *ABC*, *DEF*, about equal angles are, likewise, proportionals.

PART I.

THE ELEMENTS OF

Spherical Geometry.

SECTION IV.

ON THE RELATIVE SPECIES OF THE SIDES AND ANGLES OF A SPHERICAL TRIANGLE.

DEFINITIONS.

(120.) 1. IF a spherical triangle have one, at least, of its sides a quadrant, it is called a *Quadrantal Triangle.*

2. If a spherical triangle have one, at least, of its angles a right angle, it is called a *Right-angled Spherical Triangle.*

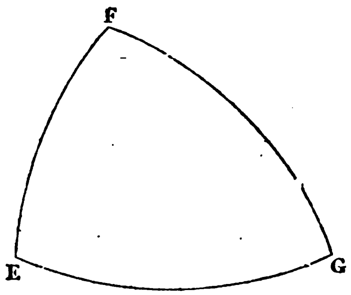
3. If a spherical triangle have none of its sides a quadrant, nor any of its angles a right angle, it is called an *Oblique-angled Spherical Triangle:*

4. **And, if each of its angles be less than a right angle, it is called an *Acute-angled Spherical Triangle.***

Prop. I.

(121.) *Theorem.* If two angles of a spherical triangle be right angles, the sides opposite to them shall be quadrants: and, conversely, if two sides of a spherical triangle be quadrants, the angles opposite to them shall be right angles.

Let *FEG* be a spherical triangle; and first, let the angles *E* and *G* be right angles: then are *FE* and *FG* quadrants.



For, (Art. 51.) *F* is the pole of *EG*, and consequently, (Art. 72. and 36.) *FE* and *FG* are quadrants.

Secondly, let *FE* and *FG* be quadrants: then, the angles *E* and *G* (Art. 37. 50.) are right angles.

(122.) Cor. 1. If all the angles of a spherical tri-

angle be right angles, all the sides are quadrants: and, if all the sides be quadrants, all the angles are right angles.

(123.) Cor. 2. Hence, it is manifest, that, on the semi-surface of a sphere, there may be as many such quadrantal, and right-angled, triangles, as there are quadrants in the great circle, which bounds that surface, and no more: wherefore, four such triangles are exactly equal to half of the surface, and eight to the whole surface, of the sphere.

Prop. II.

(124.) *Theorem.* In a right-angled spherical triangle, if either of the sides containing a right angle be a quadrant, the hypotenuse of that right angle shall also be a quadrant.

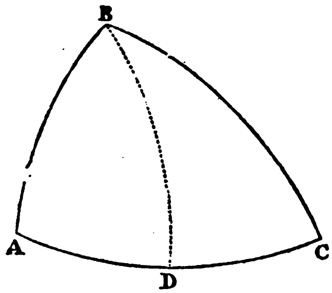
For, (Art. 50. and 36.) the extremity of the side which is a quadrant is a pole of the great circle, the arch of which constitutes the other side of the triangle: wherefore, (Art. 36.) the third side, namely, the hypotenuse, is a quadrant.

Prop. III.

(125.) *Theorem.* In a right-angled spherical triangle, if the hypotenuse of a right angle be a quadrant, one of the two sides, containing that right angle, shall also be a quadrant; and one other angle a right angle.

Let *BAC* be a right-angled spherical triangle, and

let the side BC, opposite to the right angle A, be a quadrant. Then, either AB, or AC, is a quadrant; and either C or B a right angle.



For, from C as a pole, at the distance CB, describe the circle BD, which (Art. 36.) is a great circle; and let it cut CA in D: then, if BD pass through A, it is evident, that CA (Art. 36.) is a quadrant; but if not, the angle ADB (Art. 50.) is a right angle: and the angle BAD is, by the hypothesis, a right angle; wherefore, (Art. 51.) B is the pole of AC: BA is, therefore, (Art. 36.) a quadrant, and (Art. 121.) the angle BCA is a right angle.

(126.) Def. If two sides of a spherical triangle be each of them quadrants; or if each of them be greater, or each less than a quadrant; or, if two angles of a spherical triangle be each of them a right angle; or each greater, or each less than a right angle; they are said to be *of the same species:* in all other cases they are said to be *of different species.*

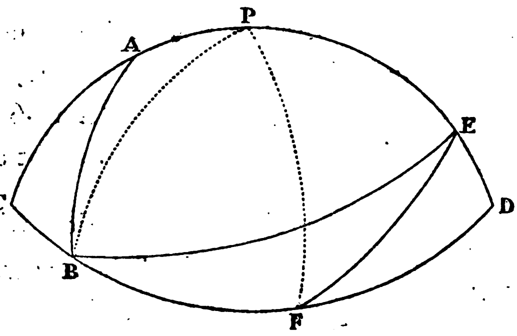
A side, also, which is a quadrant, or greater, or less,

than a quadrant, is said to be of the *same species* as the angle, which is a right angle, or greater, or less than a right angle, respectively: and in all other cases, a side and an angle are said to be of *different species*.

Prop. IV.

(127.) *Theorem.* In a right-angled spherical triangle, which has only one right angle, the two sides containing that angle are, each, of the same species as the angle opposite to it.

Let ACB be a right-angled spherical triangle, having the angle C, and no other angle, a right angle: then is the side AC of the same species as the angle B, and CB is of the same species as the angle A.



For, produce (Art. 65.) CA and CB, until they meet in D, having first found (Art. 64.) the pole P of CB; which point, because the angle C is a right angle, will (Art. 50.) be in CAD; and join (Art. 66.) P, B: where-

fore, (Art. 36.) PC and PB are quadrants, and (Art. 121.) the angle PBC is a right angle.

It is manifest, then, that accordingly as CA is less, or greater, than the quadrant CP, the opposite angle ABC will be less, or greater, than the right angle PBC; that is, CA and the opposite angle (Art. 124.) are of the same species.

And, in the same manner, may the side CB be shewn to be of the same species, as the opposite angle CAB.

(128.) Cor. The proposition is evidently true also, when it is applied to the polar triangle of a right-angled spherical triangle; that is, it is true of quadrantal triangles.

Prop. V.

(129.) *Theorem.* In a spherical triangle, the greater angle is subtended by the greater side: And, conversely, the greater side has the greater angle opposite to it.

First, let the angle ABC*, of the triangle ABC, be greater than the angle A. The side AC is greater than BC.

For, at the point B, in AB, make (Art. 96.) the angle ABG equal to the angle A: wherefore, (Art. 105.)

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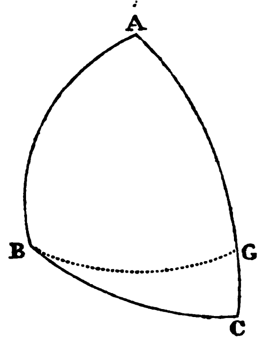
* See the figure in the next page.

GB is equal to *GA*: add to both *GC*, and *GB* and *GC*



are, together, equal to *AG* and *GC*, that is, to *AC*: but (Art. 74.) *BG* and *GC* are, together, greater than *BC*; wherefore, *AC* is greater than *BC*.

Secondly, let *AC* be greater than *BC*; then is the angle *ABC* greater than the angle *A*. For if not, it is either equal to *A*, and then (Art. 105.) *AC* is equal to *BC*; or it is less than *A*, and then, as hath been shewn, *CB* is greater than *AC*; in either of which cases, there is an absurdity: because, by the hypothesis, *AC* is greater than *BC*.

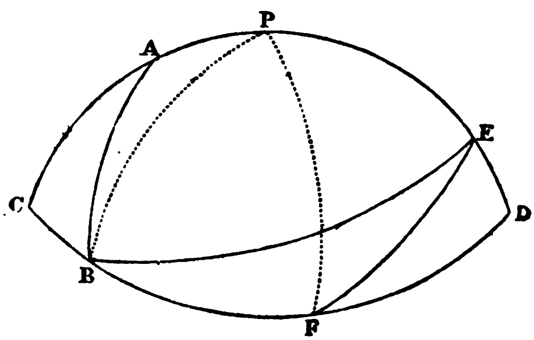
Prop. VI.

(130.) *Theorem*. In a right-angled spherical triangle, which has only one right angle, the hypotenuse of the right angle is less or greater than a quadrant, accordingly as the other sides, or the other angles, or either of the sides and the oblique angle adjacent to it, are of the same, or different species: and the converse of this proposition is also true.

Let the angle *C*, and no other angle, of the spherical triangle *ACB* be a right angle: the hypotenuse, *AB*, is less or greater than a quadrant, accordingly as *AC* and *CB*, or the angles *CAB* and *CBA*, or *CA* and the angle *CAB*, or *BC* and the angle *CBA*, are of the same or different species: and, conversely.

The same construction having first been made, as in Art. 127, take any points *E* and *F*, in *PD* and *BD*, so that *CAE*, and *CBF* may be, each of them, greater than a quadrant, and join (Art. 66.) *E*, *B* and *E*, *F* and *P*, *F*.

First, let *CA* and *CB* be each of them less than a quadrant: therefore, (Art. 127.) the angle *CAB* is less



than a right angle, and consequently, (Art. 42.) the angle *BAP* is greater than a right angle: and since, by the hypothesis, *CB* is less than a quadrant, and that *P* is the pole of *CBD*, the angle *CPB*, or *APB*, is (Art. 54.) less than a right angle: wherefore, (Art. 129.) *AB* is less than *PB*, which (Art. 36.) is a quadrant.

Secondly, let *ECF* be a spherical triangle, having the angle at *C*, and no other angle a right angle, and let the sides *EC*, *CF* be each of them greater than a quadrant; the hypotenuse *EF* is less than a quadrant: for it is also the hypotenuse of the triangle *EDF*, right-angled (Art. 56.) at *D*, and not at *E* or *F*; for, then, either *DF* or *DE* would (Art. 121.) be a quadrant: therefore, since, by the supposition, *DE* and *DF* are, each of them, less than a quadrant, *EF* is also, from the first case, less than a quadrant.

Next, let *ECB* be a spherical triangle, having the side *EC* greater, and the side *CB* less, than a quadrant: then is the hypotenuse *EB* greater than a quadrant.

For (Art. 129.) the angle *CEB*, or *PEB*, is less than the right angle *ECB*: and, since *BD* is greater than a quadrant, and that *P* is the pole of *CBD*, the angle *BPD* is greater (Art. 54.) than a right angle: wherefore (Art. 129.) *EB* is greater than the quadrant *PB*.

And, since, in all the three right-angled triangles *ACB*, *ECF* and *ECB*, the angles (Art. 127.) are of the same species as the opposite sides, it is evident, that, in each case, the hypotenuse is less, or greater, than a quadrant, accordingly as the two angles, opposite to the other sides, in that case, are of the same, or of different species.

Conversely, let the hypotenuse *AB*, or *EF*, be less than a quadrant. The two sides of the triangle are then of the same species.

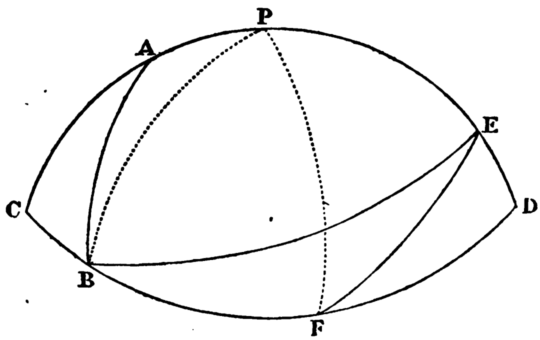
For, if not, it has been shewn that the hypotenuse is greater than a quadrant, which is contrary to the supposition.

And, in the same manner, may the remaining cases of the converse proposition be proved, *ex absurdo*.

(131.) Cor. That which has been demonstrated, in Art. 130. of a right-angled spherical triangle, must necessarily be true of its polar triangle, which is a quadrantal triangle.

Prop. VII.

(132.) *Theorem.* Accordingly, as the aggregate of any two sides of a spherical triangle is greater, or less,



than two quadrants, the angles at the base are, together, greater or less than two right angles: and the converse of this proposition is also true.

First, let *CEB* be a spherical triangle, having the

two sides *CE* and *EB*, together, greater than two quadrants: the angles *CBE*, *ECB* are, together, greater than two right angles:

For, let *CE* and *CB*, produced, meet in *D*: wherefore, (Art. 7.) *CAED* is equal to two quadrants; since, therefore, *CE* and *EB* are greater than two quadrants, they are greater than *CE* and *ED*; take away *CE* from both, and there remains *BE* greater than *ED*; therefore, (Art. 129.) the angle *D* is greater than the angle *EBD*; that is, (Art. 56.) the angle *C* is greater than *EBD*; add to both the angle *EBC*; and the two angles *ECB*, *EBC* are greater than the two *EBD*, *EBC*; that is, (Art. 42.) greater than two right angles.

Next, let the two sides *CA*, *AB*, of the spherical triangle *ACB*, be, together, less than two quadrants; then, the same construction having been made, as in the preceding case, it may be shewn, by the same mode of reasoning, and by the very same previous propositions, that the two angles *ACB* and *ABC* are, together, less than two right angles.

Conversely: first, if the two angles *ECB* and *EBC*, of the spherical triangle *ECB*, be greater than two right angles, they are greater (Art. 42.) than the two *EBC* and *EBD*: take away the common angle *EBC*; and *ECB* is greater than *EBD*; that is, (Art. 56.) the angle *D* is greater than *EBD*: therefore, (Art. 129.) *EB* is greater than *ED*; add to both *EC*, and the two *BE*, *EC* are, together, greater than *DE*, *EC*; that is, greater (Art. 7.) than two quadrants.

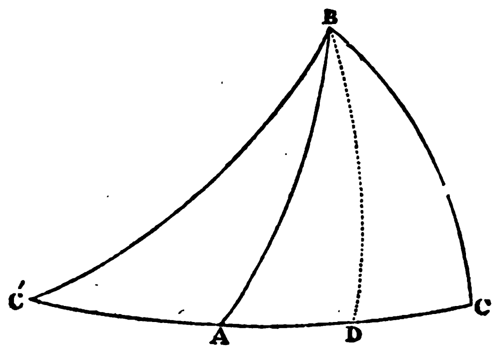
And, if the two angles *ACB* and *ABC*, of the spherical triangle *ACB*, be, together, less than two right angles, it may be proved, in like manner, that the aggregate of the two sides *CA*, *AB* is less than two quadrants.

(133.) Cor. The base of a spherical triangle having been produced, the exterior angle will be greater than the interior opposite angle, if the aggregate of the other two sides be less than two quadrants: And, if the aggregate of those other two sides be greater than two quadrants, the exterior angle will be less than the interior opposite angle: and, conversely.

Prop. VIII.

(134.) *Theorem.* The angles at the base of a spherical-triangle are of the same, or different species, accordingly as an arch of a great circle, drawn from the vertex at right angles to the base, falls within or without the base: and the converse proposition is also true.

Let *ABC* be a spherical triangle, having *AC* for its



base; and let BD be an arch of a great circle, perpendicular to AC: if BD fall within AC, the angles A and C are of the same species: but if BD fall without AC, the angles A and C are of different species: and, conversely.

First, let BD fall within AC: then (Art. 127.) in the two-right angled triangles BDA, BDC, the angle A, and the side BD, are of the same species, as also are BD and the angle C: wherefore the angles A and C are both of the same species.

Next, let ABC' be a spherical triangle, and let the perpendicular arch BD fall without AC': and since the angle BAD in the triangle BAD is of the same species (Art. 127.) as the side BD, the angle BAC' (Art. 42. and 126.) is not of the same species as BD: but (Art. 127.) the angle C' is of the same species as BD: therefore, the angles BAC' and $BC'A$, of the spherical triangle $BC'A$, are of different species.

Lastly, it is manifest, that if the converse of the proposition be not true, neither can the proposition itself be true: but it has been demonstrated to be true: therefore, its converse is also true.

(135.) Cor. The three sides, of an acute-angled spherical triangle, are, each of them, less than a quadrant.

For, it is evident, that in this case, the perpendicular arch, drawn from *any* one of the angles, will fall within the base: Thus, if all the angles of the triangle ABC be

acute angles, the perpendicular arch BD, drawn from B, will fall within AC, and the two angles DBA, DAB, of the right-angled triangle BDA, will, both of them, be acute: wherefore, (Art. 130.) AB is less than a quadrant: and, in the same manner, may any other side, of the acute-angled triangle ABC, be shewn to be less than a quadrant.

Prop. IX.

(136.) *Theorem.* If from the ends of the side of a spherical triangle, there be drawn to a point, within the triangle, two arches of great circles, they shall be, together, less than the other two sides, of the triangle.

The proposition is proved, by the help of Art. 74. exactly in the same manner, as the twenty-first proposition of the first Book of Euclid's Elements.

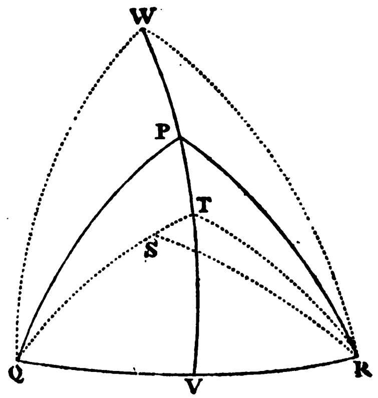
Prop. X.

(137.) *Theorem.* The side of an isosceles spherical triangle, is less or greater than a quadrant, accordingly as the angles, at the base, are acute, or obtuse angles: and, conversely.

First, let TQR be an isosceles spherical triangle having the angles at its base TQR, TRQ acute angles; TQ, or TR, is less than a quadrant.

For, draw (Art. 70.) the arches QP, RP each at right angles to QR, and meeting in P.

Then (Art. 51.) P is the pole of QR; and PQ and PR (Art. 36.) are quadrants: And, it is manifest, since the angles TQR, TRQ are, by the hypothesis, less than



the right angles PQR, PRQ, that the point T is within the triangle PQR. Therefore, (Art. 136.) QT and TR are less than QP and PR: that is, the two equal sides of the triangle are, together, less than two quadrants: therefore, each of these equal sides is less than a quadrant.

Secondly, if the angles WQR, WRQ, at the base of the isosceles spherical triangle WQR, be obtuse angles, the same construction having been made, as before, it may be shewn, in like manner, that either side of WQR is greater than a quadrant.

Lastly, the converse proposition is necessarily true; otherwise, it is evident, that the proposition itself cannot be true.

Prop. XI.

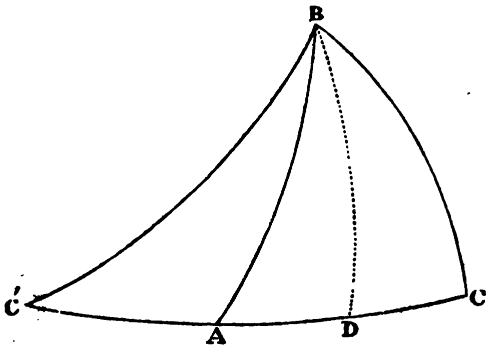
(138.) *Theorem.* If the angles at the base of a spherical triangle be unequal, but both of them acute, the side opposite to the less shall be less than a quadrant: And, if the angles at the base be unequal, but both of them obtuse, the side opposite to the greater shall be greater than a quadrant.

The proposition is proved, by a construction, and a mode of reasoning, exactly similar to those which were employed in the demonstration of Art. 137.

Prop. XII.

(139.) *Theorem.* If the angles at the base of a spherical triangle be both of them acute angles, and if a side, also, opposite to one of them be not less than a quadrant, the base shall be greater than a quadrant, and the angle opposite to it shall be obtuse.

Let the angles C and C', in the spherical triangle



$BC'C$ be both acute, and let the side $C'B$ opposite to

C be not less than a quadrant: then the angle $C'BC$ is obtuse, and the base $C'C$ is greater than a quadrant.

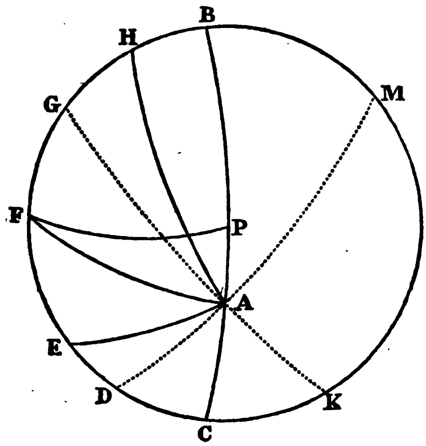
From B draw the arch BD (Art. 70.) perpendicular to the base $C'C$; which arch, because the angles C' and C are both of them acute angles, will fall (Art. 134.) within the base: and, in the right-angled triangle $C'BD$, because the hypotenuse $C'B$ is not less than a quadrant, and that the angle C' is an acute angle, therefore, (Art. 125. 130.) the angle $C'BD$ is either a right angle, or an obtuse angle: consequently, the angle $C'BC$ is, in either case, an obtuse angle.

The angle $C'BC$ of the triangle BCC', is, therefore, greater than the angle C, which, by the hypothesis, is an acute angle: therefore, (Art. 129.) CC' is greater than $C'B$, which, by the supposition, is not less than a quadrant: therefore, $C'C$ is greater than a quadrant.

Prop. XIII.

(140.) *Theorem.* If any point be taken, in a sphere's surface, which is not the pole of a given circle, of all the arches of great circles, which can be drawn from it to the circumference, the greatest is that in which is the pole, and its remainder is the least; and of the others, that which is nearer to the greatest, is greater than that which is more remote: and, from the same point, there can be drawn to the circumference, only two arches of great circles that are equal to one another, one upon each side of the shortest arch.

Let *BCD* be the given circle, *P* its pole, and *A* any other point* in the sphere's surface: also, let *CAPB* be a great circle, passing through *A* and *P* and meeting



BDC in the points *B* and *C*; and let *AD*, *AE*, be any other arches of great circles, drawn from *A* to the circumference of *BDC*, of which *AE* is nearer than *AD*, to *AB*: the greatest of the arches drawn from *A* to the circumference of *BDC* is *AB*, and *AC* is the least: and of the others *AE* is greater than *AD*.

The proof of the proposition is exactly the same as that of E. 7. 3. and follows from Art. 74. 111. 97.

(141.) Cor. 1. The greatest and least of the arches, namely, *AB* and *AC*, drawn from any point, *A*,

* If the circle *BKCD* be a lesser circle, the proposition is proved, in a similar manner, whether the point *A* be taken at a less, or at a greater distance, from the pole *P*, than the polar distance *PC*.

in the sphere's surface, to the circumference of a given great circle, *BCD*, measure the greatest and least of all the angles made by the other arches, drawn from *A* to that circumference. Also, of the rest of those angles, *AEC*, *ADC*, toward the least arch *AC*, that which is nearer to the least angle, is less than that which is more remote: and, on the other side, towards the greatest arch *AB*, the angle *AGB*, which is nearer to the greatest angle, is greater than the angle *AHB*.

For, find (Art. 64.) the pole *F* of *CAB*, which (Art. 50.) is in the circumference *BDC*; join (Art. 66.) *PF*; and the angle *CAK* having been made (Art. 96.) equal to *CAD*, produce *DA* and *KA* to meet the circumference in *M* and *G*.

Then, because *AF* is greater than *AC*, the angle *ACF* is greater (Art. 129.) than *AFC*: and *AK* (Art. 98.) is equal to *AD*, and the angle *AKC* to *ADC*: but, since *AF* is greater (Art. 140.) than *AD*, that is, than *AK*, therefore, in the triangle *AKF*, the angle *AKF* (Art. 129.) is greater than *AFC*; that is, *ADC* is greater than *AFC*.

In the same manner may *ADC* be shewn to be greater than any other angle towards *AC*, as *AEC*; and the angle *AEC* to be greater than *AFC*: wherefore the angle *AFC* is the least, and, consequently, (Art. 42.) the angle *AFB* is the greatest, of all the angles: also, (Art.

54.) since F is the pole of BAC, AC measures the least angle AFC, and AB measures the greatest angle AFB.

In the same manner, also, on the other side of AF, it may be shewn, that the angle AFB is greater than AGB, and that AGB is greater than AHB.

(142.) Cor. 2. If a point be taken, in the surface of a sphere, from which there fall more than two equal arches of great circles to the circumference of a given circle in the sphere, that point is the pole of the given circle.

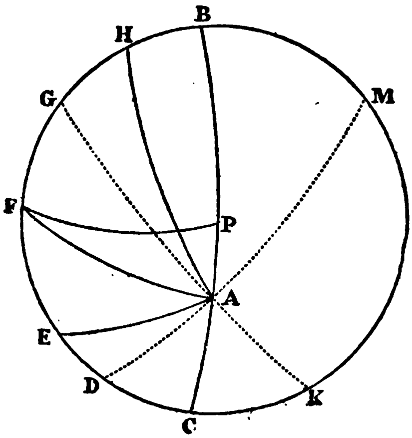
For, if it be not the pole, then from a point in a sphere's surface, which is not the pole of a given circle, there may be drawn more than two arches of great circles that are equal to one another: which (Art. 140.) is absurd.

Prop. XIV.

(143.) *Theorem.* The base of an oblique-angled scalene spherical triangle, and the opposite or vertical angle, are of the same species, if one of the remaining sides, compared with the base, and with the arch which, together with the base, makes two quadrants, is greater than the one, and less than the other: the base and the vertical angle, are also, of the same species, if one of the angles at the base, compared with the vertical angle, and with the angle which, together with the vertical angle makes up two right angles, is greater than the one, and less than the other.

Let *AED* be an oblique-angled scalene spherical triangle: let any one of its sides, as *AD*, be considered as its base; and let *DA*, produced, meet *DE*, produced, in *M*: therefore, (Art. 7.) *AM* is the arch, which together with *AD*, makes up two quadrants: also, (Art. 42.) the angles *AED* and *AEM* are, together, equal to two right angles: and, first, if one of the other sides, as *AE*, of the triangle *AED*, be greater than *AD*, and less than *AM*; *AD*, and the opposite angle *AED*, are of the same species.

For, find (Art. 64.) the pole *P* of the circle *ED*; join (Art. 66.) *P*, *A*, and produce *PA*, both ways, to meet the



circumference of *ED* in *B* and *C*: also, at the point *A* in *CA*, make (Art. 96.) the angle *CAK* equal to *DAC*; and produce *KA*, to meet the circumference of *DE* in *G*; so that (Art. 140.) *AG* is equal to *AM*, and *AE* is situated between *AD* and *AG*.

Then, it is manifest, that if *AD* be less than a quadrant, *AM*, and *AG*, are each of them greater than a quadrant; wherefore, (Art. 137.) the angle *AGM* is obtuse, and, consequently, the angle *AGD* acute: but (Art. 141.) the angle *AED* is less than *AGD*: wherefore the angle *AED* is less than a right angle, when *AD* is less than a quadrant: that is, *AD* and the angle *AED* are of the same species: and, in the same manner, if *AD* be greater than a quadrant, the angle *AED* may be shewn to be obtuse, that is, to be of the same species as *AD*.

Again, in the spherical triangle *AED*, let the angle *ADE*, at the base, be greater than the vertical angle *AED*, and less than the angle *AEM*.

Then (Art. 129.) *AE* is greater than *AD*: and, because the angle *ADE*, or (Art. 56.) *AME*, is, by the hypothesis, less than the angle *AEM*, therefore, (Art. 129.) *AE* is less than *AM*: and, therefore, by the preceding case, *AD* and the opposite or vertical angle *AED* are of the same species.

(144.) Cor. Whenever, by means of Art. 143, the *species* of two sides, and of the two opposite angles, of a spherical triangle, can be determined, it may be known, from Art. 134. whether the arch of a great circle, drawn perpendicular to the third side from the angle opposite to it, fall within, or without, that side.

PART I.

THE ELEMENTS OF

Spherical Geometry.

SECTION V.

ON THE MUTUAL CONTACT OF CIRCLES IN A SPHERE.

DEFINITION.

(145.) CIRCLES in a sphere are said to touch one another, the circumferences of which meet, but do not cut one another.

(146.) COR. Two great circles of a sphere cannot touch one another: for (Art. 7.) they bisect each other.

PROP. I.

(147.) *Theorem.* If two circles in a sphere touch one another, their common section touches each of the circles, in the point that is common to their two circumferences: and conversely.

For (E. 3. 11.) the common section of the planes of the two circles is a straight line; and it is in the plane of each of the circles; unless, therefore, it touch each of them, there will be more than one point common to the circumferences of the two circles, and therefore, (Art. 145.) they will not touch one another: which is contrary to the supposition.

The converse proposition may be shewn to be true, by a similar proof, *ex absurdo.*

(148.) Cor. 1. If two lesser circles, in a sphere, both touch the same great circle, their common sections with it are in the same plane.

For (Art. 147.) both the sections are in the plane of the great circle.

(149.) Cor. 2. Two circles, in a sphere, which touch any the same circle, of the sphere, in the same point, also touch one another.

(150.) Cor. 3. If two circles, in a sphere, touch each other, a straight line which touches either of them, in their point of contact, also touches the other.

For two planes cannot cut each other in more than one straight line; nor can more than one straight line touch a circle in the same point, of its circumference, the tangent straight line being supposed to be in the same plane with the circle.

PROP. II.

(151.) *Theorem.* If two given circles, in a sphere, meet each other in any the same point of the circumference of the great circle in which are their poles, they shall touch one another.

For (Art. 27.) their planes are both of them perpendicular to the great circle in which are their poles; wherefore, (E. 19. 11.) the common section of their planes is perpendicular to the plane of that great circle; and therefore, (E. 3. Def. 11.) it is perpendicular to the two diameters, drawn from their point of concourse, one in each of the given circles; for these diameters are in the plane (Art. 28.) of the great circle: and, since the common section of the two given circles is perpendicular to a diameter in each, it (E. 16. 3. Cor.) touches them both; wherefore, (Art. 147.) the two circles touch one another.

The proposition may, also, easily be proved *ex absurdo.*

(152.) COR. If two circles in a sphere meet each other, and if an arch of a great circle, drawn from the pole of either of them, to the point in which they meet, be perpendicular to the circumference of the other, the two circles shall touch one another.

For (Art. 50.) the poles of any circle are in the arch of a great circle, which cuts its circumference at right angles. It is manifest, therefore, that the poles of both

the circles, described in the corollary, will lie in a great circle passing through the point, in which they meet. They, consequently, (Art. 151.) touch one another.

Prop. III.

(153.) *Theorem.* If two circles in a sphere touch one onother, the arch of a great circle, which joins their poles, passes through their point of contact: and the arch of a great circle, which joins the pole of either of the circles and the point of contact, passes through the pole of the other circle.

The proposition is proved, by the help of Art. 66. and 74, exactly in the same manner, as the twelfth proposition in the third book of Euclid's Elements.

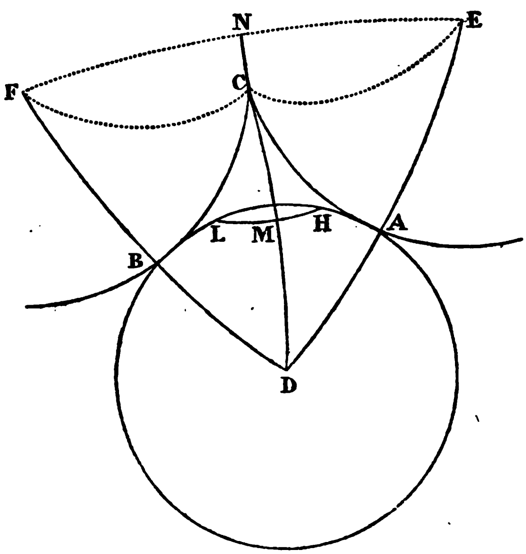
(154.) Cor. If a lesser circle, in a sphere, be touched by a great circle, the great circle shall be cut perpendicularly, by the arch of another great circle, joining the pole of the lesser circle and the point of contact.

For (Art. 151.) that arch, being produced, passes through the pole of the great circle; because the distance between the two poles being less than the semi-circumference of a great circle, there can be (Art. 16.) but one arch of a great circle which joins them ; namely, the arch joining the lesser circle's pole, and the point of contact ; which, because it passes through the pole of the great circle, is (Art. 50.) perpendicular to the circumference of that circle.

Prop. IV.

(155.) *Theorem.* If a circle in a sphere touch two equal circles, which meet each other, the two arches of the latter circles, between the point in which they meet and the two points of contact, shall be equal to one another.

Let the circle AB be touched, in the points A and B,



by the two equal circles CA and CB, which meet one another in C. The arch CA is equal to CB.

For, find (Art. 64.) the poles D, E, and F, of the three circles: and join (Art. 66.) C, D, and C, E and C, F and E, D and F, D, by arches of great circles: then (Art. 153.) ED passes through A, and FD through B.

Therefore, (Art. 27. 28.) the great circles *DB* and *DA* are at right angles to *CB* and *CA*, respectively : and *CB* and *CA* are, by the hypothesis, equal circles : wherefore, (Art. 40.) the arch *CA* is equal to the arch *CB*.

PROP. V.

(156.) *Theorem.* From any point on a sphere's surface, two equal arches of equal circles having been drawn, to meet a given circle of the sphere, if one of the arches touch the given circle, the other touches it also.

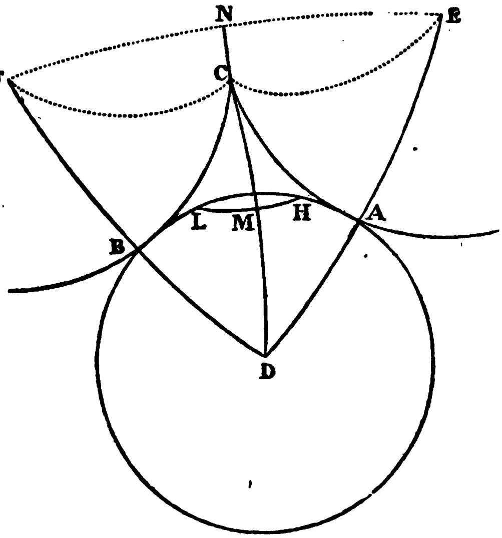
From the point *C**, let there be drawn *CA*, *CB*, two equal arches of equal circles, meeting the circle *AB* in *A* and *B* respectively : and let one of them as *CA* touch *AB* in *A*; then shall *CB* also touch *AB* in *B*.

For, the poles, *D*, *E*, and *F*, of the several circles having been found, as in the preceding proposition, and *D*, *A* and *D*, *B* having been joined, if *CA* and *CB* be great circles, the three sides of the spherical triangle *CAD* are equal to the three sides of *CBD*, each to each; because, *D* being the pole of *AB*, the arch *DA* is equal (Art. 31.) to *DB*; and *CB* is supposed equal to *CA*, and *CD* is common to the two triangles : wherefore, (Art. 86.) the angle *CBD* is equal to *CAD*; but (Art. 154.) *CAD* is a right angle; consequently, *CBD* is a right angle, and (Art. 152.) *CB* touches *AB* in *B*.

But, if *CB* and *CA* be equal arches of equal lesser

* See the figure in Art. 155.

circles, of which *E* and *F* are the poles, as before, the arch *DE* shall pass through *A*.



Suppose *C, A* and *C, B* and *F, B* to be joined by arches* of great circles: then since the sides *FB* and *EA* of the isosceles spherical triangles *ECA, FCB* are equal to one another, and since (Art. 54.) the angle *CFB* is equal to the angle *CEA*, therefore, the angle *CAE* is equal to *CBF* (Art. 97.), and the side *CB* to *CA*. Hence, in the two spherical triangles *CAD, CDB*, the three sides of the one are equal to the three sides of the other, each to each; therefore, (Art. 86.) the angle *CBD* is equal to the angle *CAD*; and the angle *CBF* has been shewn to be equal to the angle *CAE*; therefore, the two angles *CBD, CBF* are equal to the two *CAD, CAE*, that is, (Art. 154.) to two right angles: therefore, (Art. 48.) *FB* and *BD* are in one and the same circumference of a great

* These arches are not actually drawn in the figure.

circle: but *F* and *D* are the poles of the circles *CB* and *AB*: therefore, (Art. 151.) *CB* and *AB* touch one another in the point *B*.

Prop. VI.

(157.) *Theorem.* An arch which is the third part of the circumference of a great circle, in a sphere, being given, if from each of its extremities as a pole a lesser circle be described passing through the other extremity, the two cireles so described shall touch one another.

For, each of the lesser circles, so described, will have its polar distance equal to the third part of the circumference of the great circle; and therefore, they will each of them meet that circumference in the same point, namely, in the bisection of the remaining two-thirds of the circumference: consequently, (Art. 151.) they touch one another.

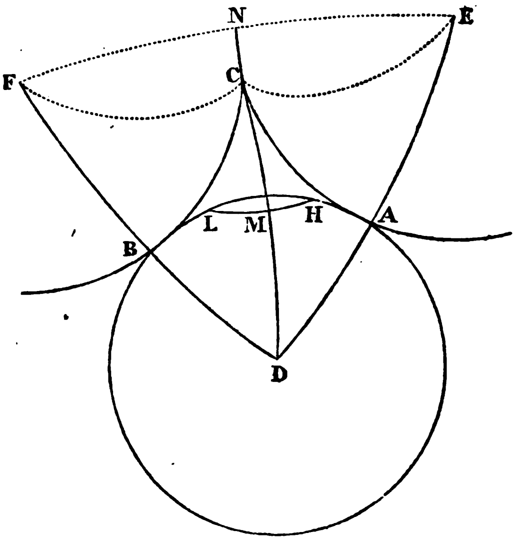
(158.) Cor. It is evident that, if the given arch be greater than a third part of the circumference of a great circle, the two circles, described, each from one of its extremities as a pole, and passing through the other extremity, will not meet each other: and that, if the given arch be less than a third part of a circumference, the two circles so described will cut one another.

Prop. VII.

(159.) *Problem.* In a given sphere, to describe a circle, which having its polar distance equal to a given

arch, shall pass through a given point on the sphere's surface, and touch another given circle.

Let *AB* be the given circle, and if the given point be in its circumference, let it be *A*: then, if the pole *D* of



AB be found (Art. 64), and *D*, *A* being joined (Art. 66.), if *DA* be produced, and *EA* made equal (Art. 92.) to the given polar distance, it is manifest, (Art. 151.) that a circle described from *E* as a pole, at the distance *EA*, will touch *AB* in the given point *A*.

But, let *C* be the given point, and let *C* be without the given circle *AB*.

It is required to describe a circle, which, having its polar distance equal to a given arch, shall pass through *C*, and touch the circle *AB*.

Find (Art. 64.) a pole, D, of AB; and from D draw (Art. 93.) an arch equal to the aggregate of the polar distances of the circle AB, and the circle which is required to be described. From D as a pole, at a distance equal to the arch so drawn, describe the circle EF, and from C, as a pole, at a distance equal to the given polar distance of the circle required to be described, describe another circle cutting EF in E and F, join D, E (Art. 66.) by the arch DE, cutting AB in A: lastly, from E, as a pole, at the distance EA, describe a circle AC. AC (Art. 151.) touches AB in A; and if C, E be joined, it is manifest, from the construction, EC and EA, being each equal to the same given polar distance, that the circle AC passes through the given point C.*

(160.) Cor. 1. In the same manner it may be shewn, that a circle described from the point F, as a pole, at the distance FB, also touches the given circle, and passes through the given point C.

(161.) Cor. 2. When therefore, the given point is without the given circle, there may be drawn two arches of great circles in the sphere, each passing through the given point, and touching the given circle: but it is manifest, from the construction, that *only two* arches can be drawn to touch a given circle in a sphere, from a point, in the sphere's surface, without the circle.

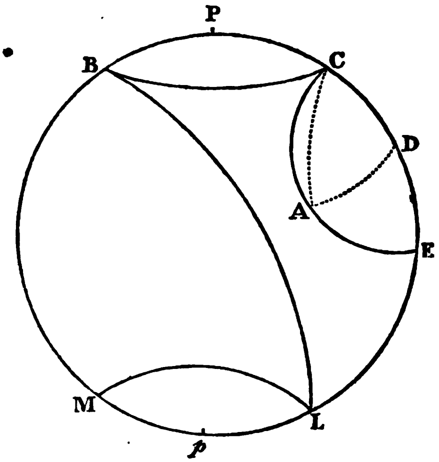
(162.) Scholium. The given point and the pole of the given circle being joined, it is plainly necessary that

the given polar distance be not less than the half of that part of the arch so drawn, which is intercepted between the given point, and the circumference of the given circle.

Prop. VIII.

(163.) *Theorem.* If a great circle touch a given lesser circle in a sphere, it shall also touch another circle, in the sphere, that is equal and parallel to the given lesser circle.

Let the great circle *BL* touch the lesser circle *BC*, in



the point *B*: it shall also touch another lesser circle of the sphere, that is equal and parallel to *BC*.

Find (Art. 64.) the poles *P* and *p*, of *BC*, and also a pole *M* of *BL*: join *P*, *M* (Art. 66.); then (Art. 153. and 23.) the great circle *PM* passes through *B* and *p*.

But (Art. 21. and 7.) *PBp* is a semi-circumference,

as also is *BML*: take away from these equals the common part *BMp*, and there remains *PB* equal to *pL*. Therefore, (Art. 25. and 38.) a circle described from *p*, as a pole, at the distance *pL*, will be equal and parallel to *BC*, and will also (Art. 151.) be touched by *BL* in *L*.

Prop. IX.

(164.) *Problem.* To describe a circle, which shall pass through a given point on a sphere's surface, and touch another given circle of that sphere, in another given point.

Let *BC** be the given circle; *C* a given point in its circumference, and *A* another given point on the sphere's surface; it is required to describe a circle of the sphere which shall pass through *A*, and touch the circle *BC* in *C*.

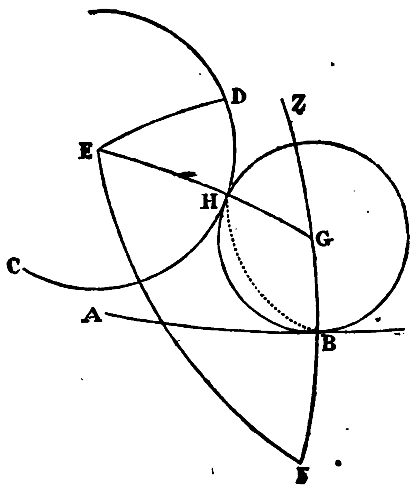
Find (Art. 64.) a pole, *P*, of *BC*; and (Art. 66.) describe the great circle *PBpC*, passing through *P* and *B*: also join *C*, *A* by the arch of a great circle *CFA*; and at the point *A*, in *CFA*, make the spherical angle *CAD* equal (Art. 96.) to *ACD*: lastly, from *D*, as a pole, at the distance *DC*, describe the circle *CE*. Then (Art. 151.) *CAE* touches *BC* in *C*; and, since the angle *DCA* is equal to *DAC*, *DC* (Art. 105.) is equal to *DA*; wherefore, the circle *CAE* passes, also, through *A*.

* See the figure in Art. 163.

Prop. X.

(165.) *Problem.* Two circles in a sphere being given, to describe another circle of that sphere, which shall touch one of the given circles in a given point, and shall also touch the other given circle.

Let *AB* and *CD* be the two given circles, and *B* a given point in one of them, namely, in *AB*: it is required to describe a circle of the sphere, which shall touch *AB* in *B*, and shall also touch the circle *CD*.



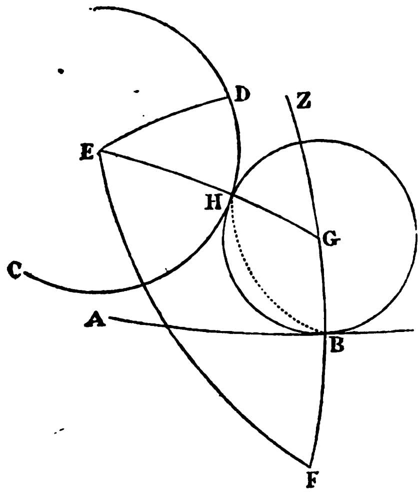
Find (Art. 64.) a pole *E*, of *CD*: from *B* draw (Art. 70.) an arch of a great circle *BZ*, at right angles to *AB*; produce (Art. 65.) *BZ* to *F*, and make (Art. 92.) *BF* equal to *ED*, the polar distance of the circle *CD*. Join (Art. 66.) *E*, *F*; and at the point *E*, in *FE*, make (Art. 96.) the spherical angle *FEG* equal to *GFE*, and let *EG* meet *BZ* in *G*, and *CD* in *H*. Lastly, from *G*, as a pole, at the distance *GH*, describe the circle *HB*.

Then, *HB* (Art. 151.) touches *CD*: also, since, by the construction, the angle *E* is equal to the angle *F*, *GE* is equal to *GF* (Art. 105.): and *BF*, by the construction, is equal to *EH*: wherefore, *GB* is equal to *GH*, and the circle *HB* passes through *B*. Again, since *BGZ* is at right angles to *AB*, the poles of *AB* (Art. 50.) are in *BZ*. And, since *HB* and *AB* meet one another in the same point of the great circle, in which are their poles, they (Art. 151.) touch one another in that point.

Prop. XI.

(166.) *Problem.* To describe a circle, which shall pass through a given point in a sphere's surface, and touch a given circle of the sphere, in another given point.

Let *B* be the given point in the sphere's surface, *DC*



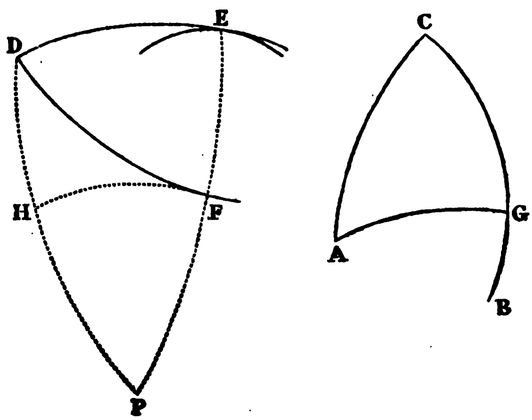
the given circle, and *H* the given point in its circumference. It is required to describe a circle which shall pass through *B* and touch the circle *DC* in *H*.

Find (Art. 64.) a pole, *E*, of *DC*; join (Art. 66.) *H*, *B*, and *E*, *H*; produce *EH* to *G*; and make (Art. 96.) the spherical angle *HBG* equal to *GHB*: lastly, from *G*, as a pole, at the distance *GH*, describe the circle *HB*: *HB* (Art. 151.) touches *DC* in *H*; and since the angle *GBH* was made equal to *GHB*, the side *GB*, of the triangle *GHB*, is equal (Art. 105.) to *GH*: wherefore, the circle *HB*, described from *G*, as a pole, at the distance *GH*, passes, also, through the given point *B*: and it has been shewn, that it touches the circle *DC*, in the other given point *H*.

Prop. XII.

(167.) *Problem.* To describe a circle, which shall have its pole in a given great circle of a sphere, and which, having its polar distance equal to a given arch, not a quadrant, shall also touch another great circle of the sphere.

Let *DE* and *DF* be two given great circles in a



sphere, and *AG* a given arch, not a quadrant. It is re-

quired to describe a circle, which, having its pole in *DF*, shall have its polar distance equal to *AG*, and shall touch *DE*.

By the method followed in Art. 117. find a point *F*, in *DF*, from which the arch *FE*, let fall at right angles to *DE*, shall be equal to *AG*.

Then, it is manifest, since (Art. 50.) the poles of the circle *DE* are in *EF*, that a circle, described from *F*, as a pole, at the distance *FE*, will (Art. 151.) touch the circle *DE*, in *E*: and its polar distance was made equal to the given arch *AG*. If therefore, (Art. 59.) a circle be described from *F* as a pole, at the distance *FE*, that, which was required, will be done.

Prop. XIII.

(168.) *Problem.* Two circles of a sphere being given, to describe a circle, of that sphere, which shall touch them both.

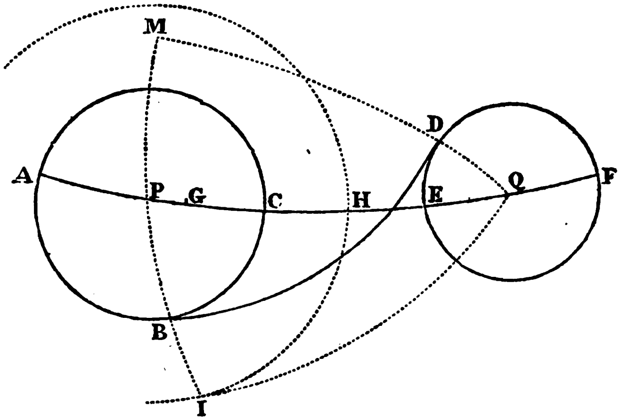
If the two given circles be equal to one another, an indefinite number of circles may be drawn, each touching them both. For, any point in the arch, bisecting, at right angles, the spherical distance of their poles, being (Art. 103.) equidistant from those poles, will be also equidistant from their circumferences.

But, let *ABC** and *DEF* be the two given circles, and let them be unequal. It is required to describe a circle, which shall touch both *ABC* and *DEF*.

Find (Art. 64.) *P*, and *Q*, the poles of the given

* See the figure in the next page.

circles, and describe (Art. 66.) a great circle, *APQF*, passing through *P* and *Q*. Then, first, it is manifest,



that a circle described from the bisection of the arch *CE*, as a pole, meeting *ABC* in *C*, and *DEF* in *E*, will touch the two given circles, in the points *C* and *E*: and, in the same manner, a circle may be described, from the bisection of *AF*, as a pole, touching the given circles in *A* and *F*: Next, from *CP*, and *CQ*, cut off *CG*, and *CH* (Art. 92.), each equal to *QE*, the polar distance of the lesser circle. From *P*, as a pole, at the distance *PH*, describe the circle *HI*. From *Q* draw (Art. 163.) an arch *QI* of a great circle touching the circle *HI*; and let it touch *HI* in *I*: join (Art. 66.) *P*, *I*, and let *PI* cut *ABC* in *B*: from *Q* draw (Art. 70.) the arch *QM* at right angles to *IQ*, cutting *DEF* in *D*, and let it meet the arch *IP*, in *M*. Therefore, (Art. 50.) *M* is the pole of *IQ*, and (Art. 31.) *MI* is equal to *MQ*: but (Art. 32.) *BI* is equal to *CH*; and *CH* was made equal to *QD*; wherefore *MB* is equal to *MD*: and the circle described from

M, as a pole, at the distance *MB*, will pass through *D*, and (Art. 151.) it will touch the circle *ABC*, in *B*, and the circle *DEF*, in *D*.

In the same manner, if an arch be drawn from *Q*, so as to touch the circle described from *P*, as a pole, at the distance *PG*, then a circle may be found which shall touch both the given circles on the *same* side; as *BD* touches them on *different* sides.

PROP. XIV.

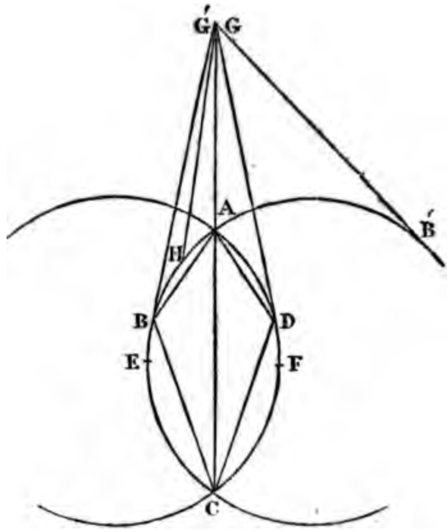
(169.) *Theorem.* If two equal circles in a sphere meet each other, and two straight lines be drawn, one touching each circle, in points that are equidistant from either of the two intersections of the circles, the tangents, so drawn, shall either be parallel to each other, or else they shall meet in the same point of the common section of the circles, and shall also be equal to one another.

Let *AB* * and *AD* be two equal arches, of the equal circles *ABC*, *ADC*, of which the common section is the straight line *CG*; and let *E* and *F* be the bisection of the arches *ABC*, *ADC*: then, it is evident, from E. 30. 3. and 1. 3. Cor. that the straight lines drawn to touch the circles, in *E* and *F*, are each parallel to *AC*, and therefore, (E. 9. 11.) they are parallel to each other.

But, let *AB* and *AD* be any other two equal arches of *ABC* and *ADC*; and first, let the circles cut each other, in the two points *A* and *C*: let *DG* and *BG'* be drawn touching them, in *D* and *B*; and since *DG* and

* See the figure in the next page.

BG' are, each of them, in the same plane as CA, and not parallel to it, they will both meet it: let them meet



CA, produced, in G and G', respectively; then, since (E. 28. 3.) the arch ABC is equal to ADC, and that AB is equal to AD, therefore, also, the arch BC is equal to DC: whence, (E. 29. 3.) the chord AB is equal to the chord AD; the angle BAC is equal (E. 27. 3.) to DAC, and the angle BCA to DCA; therefore, (E. 13. 1.) the angle BAG' is equal to DAG; and the angle $G'BA$, being equal (E. 32. 3.) to BCA, and the angle GDA to DCA, therefore the angle $G'BA$ is equal to GDA; and the chord AB has been shewn to be equal to AD: therefore, (26. 1.) the two triangles BAG' and DAG are equal to one another. So that BG' is equal to DG, and AG' to AG; that is, the tangents, drawn from B and D, are equal to one another, and meet CA produced in the same point.

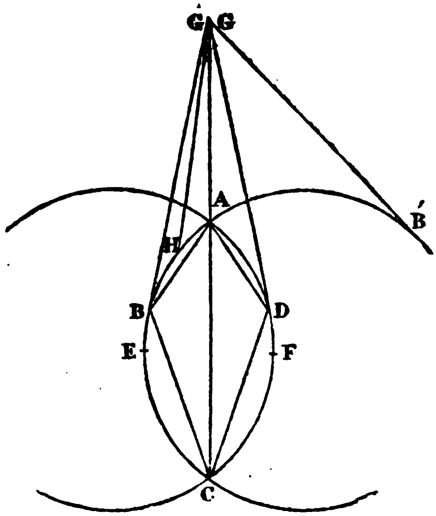
And, in the same manner, may the proposition be proved, if the two circles meet only in one point, A; that is, if they touch one another: for then (Art. 147.) the common section of their planes, AG, will touch them both; and it follows, from E. 36. 3. that $G'B$ is equal to $G'A$, and GD to GA; wherefore, (E. 5. 1.) the angle $G'AB$ is equal to $G'BA$, and the angle GAD to GDA: but the angle $G'BA$ may be shewn, as before, to be equal to GDA; and AB to be equal to AD: therefore, AG' is equal to AG, and the two tangents meet in the same point G, of the common section of the circles: and, therefore, each being equal to the same straight line GA, they are equal to one another.

Prop. XV.

(170.) *Theorem.* If two circles in a sphere meet each other, and two straight lines be drawn, the one touching the one circle, and the other the other, the two tangents so drawn, if they be parallel, are each of them parallel to the common section of the circles, and each of them is drawn through the bisection of the arch which it touches: and if they meet, they meet in the same point of the common section of the circles, and are equal to one another.

If the two tangents be parallel, and one of them also be parallel to the common section of the circles, the other must (E. 9. 11.) be likewise parallel to that section: But if it be possible, let neither of them be parallel to the common section; therefore, since each tangent is in the same plane with that section, they must both, if produced, meet it; and consequently, (E. 7. 11.) the

two tangents, and the common section of the circles, are all in the same plane; that is, the two circles are in the same plane: which is contrary to the supposition. Wherefore, the tangents are each of them parallel to the common section; and it is evident, from E. 29. 1. 3. 3. and 30. 3. that they pass through the bisections of the arches which they touch.



But, let *ABC, ADC,* be circles, which meet each other, in a sphere, having *GC* for their common section; and let the tangents *BG, DG,* meet each other in *G*. Then *BG* and *DG,* being each in the same plane with *CA,* must both meet it; and since the three straight lines *BG, DG* and *CG* are not all in the same plane, they can only meet in one point; otherwise they would have three points of intersection, and therefore, (E. 2. 11.) they would all be in the same plane; which is contrary

to the supposition. Therefore, *BG*, *DG*, and *CG* all meet in the same point *G*: and since (E. 36. 3.) each of the squares of *GB* and of *GD*, is equal to the same rectangle *CG*, *GA*, if the circles cut each other, or to the same square of *GA*, if they touch one another in *A*, therefore, *GB* is equal to *GD*.

(171.) Cor. 1. If the two circles *ABC*, *ADC* be equal, and the two tangents *BG*, *DG* meet each other, then the arch *AB* is equal to the arch *AD*.

For, if *AB* be not equal to *AD*, make (E. 1. 4. and 28. 3.) the arch *AH* equal to *AD*; and at *H* suppose a tangent, *HG'*, to be drawn to the circle *ABC*: wherefore, (Art. 160.) *HG'* meets *DG*, in *G*, and is equal to it: but *GD* has been proved to be equal to *GB*: therefore, *GH* is equal to *GB*, which (E. 8. 3.) is absurd; for, it is plain, that another tangent, *GB'*, may be drawn to the circle *ABC*, on the other side of *GA*, which tangent (E. 36. 3.) is equal to *GB*.

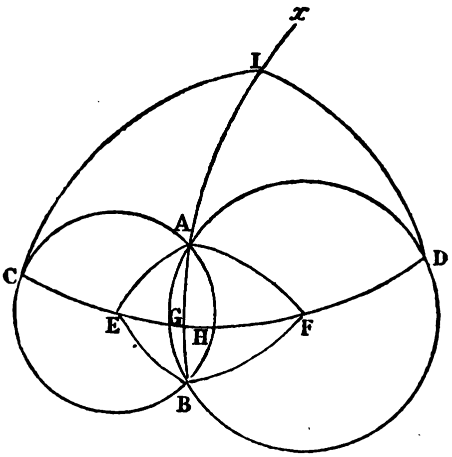
(172.) Cor. 2. From the demonstration, it is manifest, that if each of any three straight lines, which are not all in the same plane, meets both the others, the three straight lines must all meet in the same point.

Prop. XVI.

(173.) *Problem.* In the arch of a great circle, join-

ing the intersections of two given lesser circles, in a sphere, to find a point, from which, if an arch of a great circle be drawn so as to touch either of the given circles, it shall be a quadrant.

Let Bx be an arch of a great circle joining the two points of intersection A and B, of two given lesser circles in a sphere, ABC, ABD. It is required to find a point



in Bx, from which an arch of a great circle, being drawn so as to touch either ABC, or ABD, it shall be equal to a quadrant.

Find (Art. 64.) the poles E and F of ABC and ABD; join E, F (Art. 66.); and produce EF, both ways, to meet the circumferences of the given circles in C and D; from either of the two last mentioned points, as C, describe (Art. 70.) the arch of a great circle CI, at right angles to CED, and let CI meet Bx in I: I is the point required.

For, join E, A, and E, B, and F, A, and F, B. Then (Art. 31.) EA being equal to EB, and FA to FB, and EF being common to the two spherical triangles EAF, EBF, therefore, (Art. 86.) the angle AEF is equal to BEF.

Again, since the angle AEG is equal to BEG, EA equal (Art. 31.) to EB, and EG common to the triangles AEG, BEG, the angle AGE is equal to BGE; therefore, (Art. 42.) each of them is a right angle; whence (Art. 50.) the pole of EF is in GA; and it is also in CI, which was drawn at right angles to FEC: therefore, it is in I, the intersection of these arches. So that IC is a quadrant, and (Art. 152.) it touches the circle ABC in C: wherefore I is the point required. And, since I has been shewn to be the pole of CD, if I, D be joined, ID is (Art. 36.) a quadrant; and (Art. 27. 151.) ID touches the circle ABD in D.

(174.) Cor. 1. It appears, from the above proof, that the arch of a great circle, joining the poles of two given lesser circles in a sphere, which cut each other, bisects, at right angles, the arch joining their intersections; and also bisects each of the arches of the two given circles included between the points of their section: it is further evident, that if the two given circles be equal, the arch joining their intersections, bisects at right angles the spherical distance of their poles.

(175.) Cor. 2. Since (Art. 172.) C and D are the

bisections of the arches *ACB*, *ADB*, it is evident, from E. 30. 3. and 18. 3. that the straight lines which touch the given circles, in *C* and *D*, are parallel to the straight line joining *A*, *B*: and, since it is impossible for one of the tangent arches, *IC*, *ID*, to be a quadrant, and the other not a quadrant, if one of the straight lines, drawn touching the two given circles in *C* and *D*, be parallel to the chord *AB*, the other must also be parallel to it.

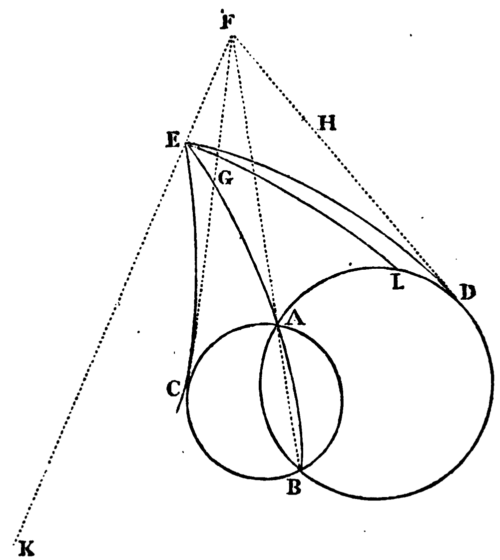
It is likewise plain, that, if arches of great circles be drawn from any other point in *Bx*, on the same side of *EF*, as *I* is, to touch the circles *ABC*, *ABD*, they will not touch the circles in the points *C* and *D*; for two great circles cannot both touch a lesser circle in the same point; because they would then touch one another, which (Art. 146.) is impossible. Therefore, the straight lines drawn touching the given circles, at these last mentioned points of contact, will not be parallel to the chord *AB*.

Prop. XVII.

(176.) *Theorem.* If two lesser circles of a sphere cut each other, and from any point in the arch of a great circle, passing through their intersections, two arches of great circles be drawn so as to touch the given lesser circles, they shall be equal to one another.

Let the two lesser circles *ABC*, *ABD* cut each other in *A* and *B*, and let *BAE*, which joins *B*, *A*, be an arch of a great circle of the sphere: if, from any point *E*, in *BAE*, two arches of great circles, *EC* and *ED*, be drawn touching *ABC* and *ABD*, in *C* and *D* respectively, the arch *EC* is equal to *ED*.

For, let K be the sphere's center; let K, E be supposed to be joined, and let KE, which is the common section of the three great circles EA, EC and ED, be



supposed to be produced to F. Suppose, also, straight lines to be drawn, touching CE and ED in C and D; they will be tangents (Art. 150.) also to the circles ABC and ABD: and if they be parallel, the arch EC (Art. 168.) is equal to ED.

But, let the two straight lines CG, and DH, drawn so as to touch the given lesser circles in C and D, be not parallel; join A, B, by the straight line AB: then, since KE is the common section of the three planes EC, EA and ED, therefore, KE and CG are in one plane, as also are KE and BA; and it is evident, that CG and

BA are both in the plane of the circle *ACB*: and, because *CG* and *DH* are, by the hypothesis, not parallel to one another, they can neither of them (Art. 170. and 175.) be parallel to *KE*, or to *AB*: therefore, *KE*, and *CG* must meet, if they be produced; as, also, must *CG* and *BA*: and these three straight lines are not all in the same plane; therefore, (Art. 170.) they must meet in one point; that is *CG* must pass through the point *F*, in which *BA* and *KE* meet: and in the same manner, it may be shewn, that *DH* produced, passes through *F*. Therefore *CG* and *DH* meet in *F*, and consequently, (Art. 169.) the arch *EC* is equal to the arch *ED*.

(177.) Cor. If an arch *EL* of a great circle, drawn from the point *E*, in *BAE*, to the circumference of *BAD*, be equal to the arch *EC* which touches the other circle, *EL* shall touch *ABD*.

For if not, draw (Art. 159.) the arch of a great circle *ED* on the same side of *EAB* as *EL*, touching the circle *ABD*. Then (Art. 176.) *ED* is equal to *EC*: and *EL* is, by the hypothesis, also equal to *EC*: therefore *EL* is equal to *ED*: which (Art. 140. note.) is absurd.

Prop. XVIII.

(178.) *Problem.* To describe a circle in a sphere, which shall pass through three given points in the sphere's surface, that are not all in the circumference of the same great circle.

The three given points having first been joined, (Art. 66.) a circle may be described about the spherical triangle, of which they are the angular points, exactly in the same manner, as a circle is described about a plane triangle, in E. 5. 4. The construction depends upon Art. 66. 102. 70; the proof follows from Art. 97.

(179.) Cor. 1. Thus may a point be found, in the surface of a sphere, that shall be equally distant from the three angular points of a given spherical triangle in that surface.

(180.) Cor. 2. The circle described about a spherical triangle is, in all cases, a lesser circle of the sphere.

For, if it were a great circle, then each of the sides of the triangle (Art. 7.) would be a semi-circumference; which (Art. 72.) is impossible.

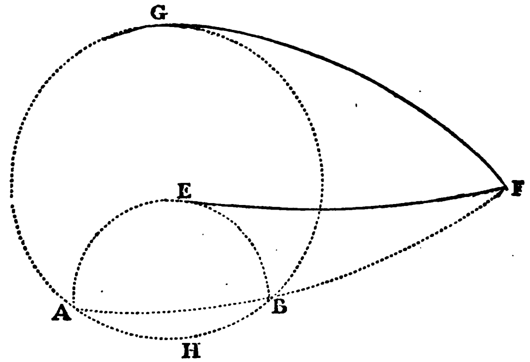
Prop. XIX.

(181.) *Problem.* Two points on the surface of a sphere being given, of which the spherical distance is less than two quadrants, to describe a circle which shall pass through them, and which shall touch a given great circle of the sphere; the two points being both of them without the given great circle.

Let *A*, *B* * be the two given points, and *GF* the given great circle: it is required to describe a circle of the sphere, which shall pass through *A* and *B*, and which shall touch *GF*.

* See the figure in the next page.

Join (Art. 66.) A, B, and produce AB to meet GF in F: take any point E, out of AB, and describe (Art.



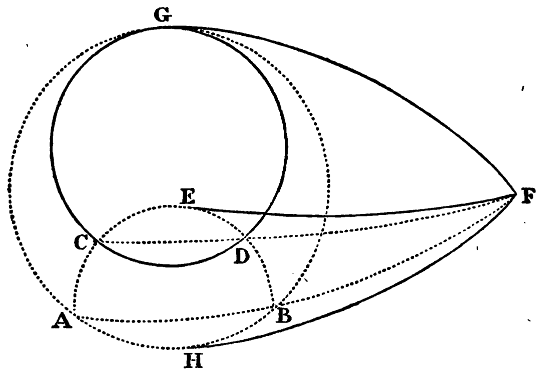
178.) a lesser circle AEB, passing through the three points A, E and B. From F draw (Art. 159.) the arch of a great circle, FE, touching AEB; and make (Art. 92.) FG equal to FE: lastly, describe (Art. 178.) a lesser circle ABG, passing through A, B and G: the circle ABG touches (Art. 177.) FG in G.

Prop. XX.

(182.) *Problem.* Two points on the surface of a sphere being given, of which the spherical distance is less than two quadrants, to describe a circle, which shall pass through them, and which shall touch a given lesser circle of the sphere; the two points being, both of them, without the given circle.

Let A, B be the two given points, and CDG the given lesser circle: it is required to describe a circle of the sphere, that shall pass through A and B, and shall touch the circle CDG.

Take any point *E*, such that *E*, and the two given points *A*, *B* are on contrary sides of the circumference of



CDG: through *A*, *E* and *B* (Art. 178.) describe the circle *AEB*, which (Art. 180.) is a lesser circle, cutting *CDG* in *C* and *D*: join (Art. 66.) *C*, *D*, and *A*, *B*; and let *CD* and *AB*, produced, meet in *F*. From *F* draw (Art. 159.) an arch of a great circle *FG*, touching the given circle *CDG* in *G*: lastly, describe (Art. 178.) a circle *GAHB*, passing through the three points *G*, *A* and *B*. The circle *GAB* passes through *A* and *B*, and touches the given circle in *G*.

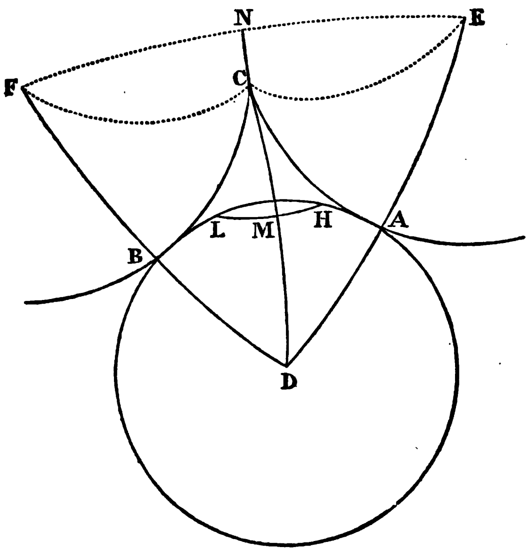
For (Art. 159.) describe the great circles *FE* and *FH*, touching the circles *AEB*, and *AHG*, which (Art. 180.) is a lesser circle, in *E*, and *H*, respectively. Then (Art. 175.) *FG* is equal to *FE*, and *FE* is equal to *FH*: wherefore *FG* is equal to *FH*: but *FH* touches the circle *GAH*; therefore, (Art. 156.) *FG* also touches the circle *GAH* in *G*: and it was drawn touching the given circle *GCD* in *G*: therefore, (Art. 149.) the circle *GAB*,

which passes through the given points *A* and *B*, touches the given circle *GCD* in *G*.

Prop. XXI.

(183.) *Theorem.* If a given great circle pass through the common section of two equal circles, in a sphere, to the planes of which it is equally inclined, any circle in the sphere, having its pole in the circumference of the given great circle, which touches one of the two equal circles, shall touch the other also.

Let the great circle *CD* pass through the common



section of the two equal circles *CA* and *CB*, to the planes of which it is equally inclined: and let the circle *AB*, of which the pole, *D*, is in *CD*, touch the circle *CA*: it shall also touch the circle *CB*.

For, find (Art. 64.) the poles, *E* and *F*, of *CA* and *CB*; join (Art. 66.) *F*, *E*, and let *DC* produced meet *EF* in *N*. Whether, therefore, the equal circles *CA* and *CB* be great circles, or lesser circles, of the sphere, *DN* (Art. 58. and 174.) bisects *EF* at right angles. Join *D*, *E*; the arch *DE* passes (Art. 153.) through the point of contact *A*: join, also, *D*, *F*, and let *DF* meet *CB* in *B*.

Then, since the sides *DN*, *NF*, of the spherical triangle *DNF*, are equal to the sides *DN*, *NE*, of the spherical triangle *DNE*, each to each; and that the angles at *N* have been shewn to be right angles, therefore, (Art. 97.) *DF* is equal to *DE*: but (Art. 38.) *FB* is equal to *EA*: therefore, *DB* is equal to *DA*; and consequently, the circle described from *D*, as a pole, at the distance *DA*, passes through *B*, and (Art. 51.) touches *CB* in that point.

(184.) Cor. If the two equal circles *CA* and *CB* touch one another, and also touch a great circle *CD*, in the same point *C*, it is manifest, from the demonstration, that any circle, in the sphere, having its pole in *CD*, which touches one of the two equal circles, will touch the other also.

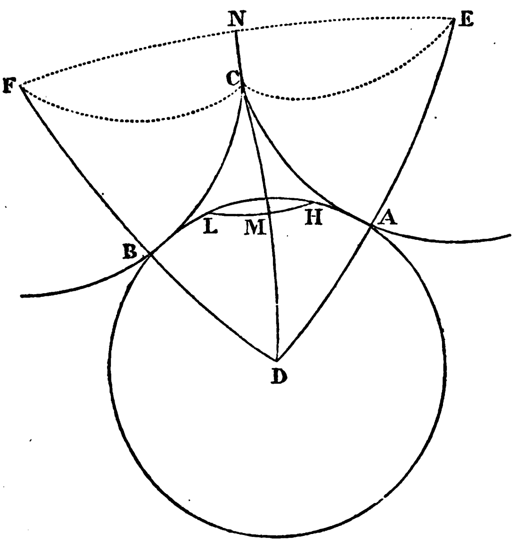
For then (Art. 153.) the arch *FE* which joins the poles *E*, *F* of the two equal circles, passes through the point of contact *C*, and also (Art. 154.) cuts the arch *DCN* at right angles: wherefore, the proof of the proposition is equally applicable to this case.

Prop. XXII.

(185.) ***Problem.*** To describe a circle which shall pass through a given point, on the surface of a sphere, and which shall touch two equal circles of the sphere, that meet one another.

If the given point be in the circumference of either of the given circles, the problem has already (Art. 165.) been solved.

But, let *CA* and *CB* be two given equal circles in a sphere, that meet in *C*; and let *H* be a point that is



not in either of them: it is required to describe a circle of the sphere, which shall pass through *H*, and shall touch both *AC* and *BC*.

If *CA* and *CB* be great circles, bisect (Art. 101.) the spherical angle *ACB*, by the arch *CD* of a great circle: but if *CA* and *CB* be equal lesser circles, which cut one another, join (Art. 66.) their intersections, by the arch *CD* of a great circle: and if they touch one another, draw (Art. 159.) the arch of a great circle, *CD*, touching either of them in *C*; and (Art. 149.) it touches the other also. Next, from the given point *H*, let fall (Art. 70.) on *CD*, the arch *HML*, cutting *CD* at right angles, in *M*; and make (Art. 92.) *ML* equal to *MH*. Lastly, describe (Art. 181. 182.) a circle *AHLB*, that shall pass through the points *H* and *L*, and touch the circle *CA*. It shall, also, (Art. 183, 184.) touch the circle *CB*.

Prop. XXIII.

(186.) *Problem.* To describe a circle in a sphere, which shall pass through a given point on the sphere's surface, which shall have its pole in a given great circle, and shall also touch another given circle of the sphere.

Let *H** be a given point on a sphere's surface; let *CD* be a given great circle, and let *CA* be any other given circle. It is required to describe a circle of the sphere which shall pass through *H*, which shall have its pole in *CD*, and which shall touch *CA*.

From *H* let fall (Art. 70.), on *CD*, the arch *HML*, cutting *CD*, at right angles, in *M*, and make (Art. 92.) *ML* equal to *MH*. Describe (Art. 181, 182.) a circle *LHA*, passing through *L* and *H*; and touching *CA* in

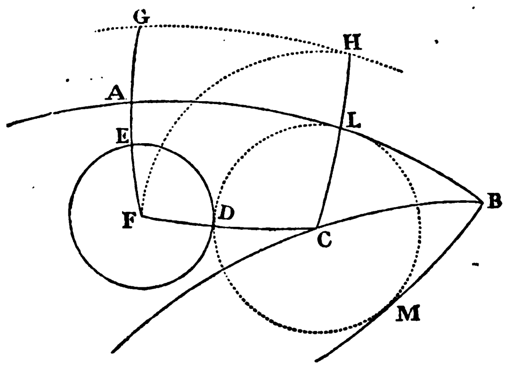
* See the figure in Art. 185.

A. The pole of the circle *LHA* is in the given great circle *CD.*

For (Art. 31.) the pole of the circle *LHA* is equidistant from the two points *L* and *H,* in its circumference: and since, by the construction, the arch *CMD* bisects *LH* at right angles, no point which is out of *CD* (Art. 104.) can be equidistant from *L* and *H.* Wherefore, the pole of the circle *LHA* is in *CD.*

Prop. XXIV.

(187.) *Problem.* To describe a lesser circle, in a sphere, which shall touch both a given great circle, and a given lesser circle of that sphere, and shall also have its pole in another given great circle.



Let *BA* and *BC* be two given great circles, cutting one another in *B*; and let *DE* be a given lesser circle, in the same sphere: it is required, to describe a circle of the sphere, which shall have its pole in *BC,* and shall touch both the other given circles *AB* and *DE.*

Find (Art. 64.) the pole *F* of *DE*; from *F* describe (Art. 70.) the arch *FA*, at right angles to *AB*, and meeting *AB* in *A*: make (Art. 92.) *AG* equal to *FE*: Find, also, the pole of *AB*, and from that point, as a pole, describe a lesser circle, *GH*, passing through *G*. Describe a circle *FH*, (Art. 186.) having its pole in *BC*, passing through *F*, and touching the circle *GH*; and let it touch *GH* in the point *H*: from *H* draw the arch *HLC*, cutting *AB* at right angles in *L*, and meeting *BC* in *C*. Then will the circle *LDM*, described from *C* as a pole at the distance *CL*, touch (Art. 152.) *AB* in *L*; and it will also touch the circle *ED*.

For, join (Art. 66.) *F*, *C*, by the arch of a great circle, *CF*, meeting the circumference of *ED*, in the point *D*.

Then, since by the construction, the pole of *FH* is in *BC*, and since (Art. 25. 49. 50.) it is also in *HC*, therefore, *C* is the pole of *FH*; and (Art. 31.) *CF* is equal to *CH*: again, (Art. 32.) *GA* is equal to *HL*; but *GA* was made equal to *FE* or *FD*; therefore, *HL* is equal to *FD*: and *CH* has been shewn to be equal to *CF*; wherefore, *CD* is equal to *CL*, and the circle *LD*, described from *C*, as a pole, at the distance *CL*, meets the circle *ED* in *D*, and also (Art. 151.) touches that circle in *D*.

PROP. XXV.

(188.) *Problem.* To describe a circle in a sphere, which shall touch two given great circles, and which shall also touch another given lesser circle, of that sphere.

Let *AB** and *BM* be two given great circles of a sphere, cutting one another in *B*, and let *ED* be a given lesser circle of the sphere: it is required to describe a circle of the sphere, which shall touch all the three given circles *AB*, *BM* and *ED*.

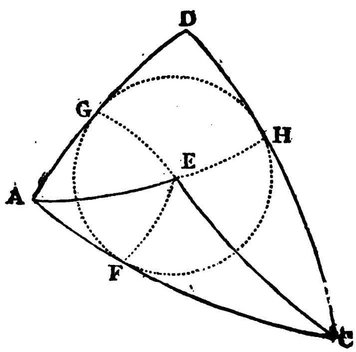
Bisect the angle *ABM* (Art. 101.) by the arch of a great circle *BC*: describe (Art. 186.) a circle *LD*, which shall have its pole in *BC*, and shall touch both the circles *ED* and *AB*: it shall also (Art. 183.) touch the circle *BM*.

Prop. XXVI.

(189.) *Problem.* To describe a circle in a sphere, which shall touch the three sides of a given spherical triangle.

Let *ADC* be a given spherical triangle: it is required to describe a circle, which shall touch its three sides.

Bisect (Art. 101.) the angles *A* and *C*, by the arches



* See the figure in Art. 187.

AE, and *CE*; and let *AE* and *CE* meet in *E*: describe (Art. 70.) the arch *EF*, at right angles to *AC*. Then, a circle described from *E*, as a pole, at the distance *EF*, will (Art. 152.) touch *AC*; and, since *AE* and *CE* bisect the angles *A* and *C*, it will (Art. 183.) touch *AD* and *CD*.

(190.) COR. 1. Hence, two points on the surface of a sphere may be found, each equidistant from the sides of a given triangle on the sphere, one on the one side of the base, and the other on the other.

(191.) COR. 2. It is evident, from the demonstration, and from Art. 116, that the three arches of great circles, which bisect the three angles of a spherical triangle, all meet in the same point of the sphere's surface: and that the three arches of great circles drawn from that point perpendicular to the three sides of the triangle, each to each, are equal to one another.

(192.) SCHOLIUM. A great variety of problems of the same kind, as those which have been treated of in this section, might still be added. But after the examples which have already been given, it is presumed, that the student will have no difficulty in solving such of them, as his curiosity shall lead him to consider. The two propositions last demonstrated are, in reality, particular cases of this general problem; "Three circles being given, in a given sphere to describe a fourth circle

of the sphere, which shall touch the other three." The remaining cases are not more difficult, than those which have been exhibited: and the problem may be solved, generally, exactly in the same manner, as a fourth circle is described, so as to touch three given circles, when all the circles are in the same plane.

PART I.

THE ELEMENTS OF

Spherical Geometry.

SECTION VI.

ON THE COMPARISON OF SPHERICAL SURFACES WITH ONE ANOTHER.

Prop. I.

(193.) *Theorem.* Two portions of a sphere's surface, which are contained by equal arches of equal lesser circles, are equal to one another.

For, since the arches, which bound the two figures, are equal to one another, if the one figure be applied to the other, as it evidently may be, so that one of the equal arches which bound it, may coincide with one of the arches bounding the other, the two figures shall wholly coincide: otherwise, three equal circles in a sphere might have a common section; which (Art. 14.) is absurd:

since, therefore, the two figures, when the one is applied to the other, wholly coincide, they are equal to one another.

Prop. II.

(194.) *Theorem.* Two portions of a sphere's surface, which are contained by the semi-circumferences of great circles, are equal to one another, if the spherical angle, contained by the one pair of semi-circumferences, is equal to the spherical angle contained by the other.

For, if the one figure be applied, as it evidently may be, to the other, so that either of the arches which bound it, may coincide with either of the arches, which bound the other, it is plain that the two figures will wholly coincide, and will, therefore, be equal to one another.

Prop. III.

(195.) *Theorem.* If the two extremities of any arch of a lesser circle, in a sphere, be joined by the arch of a great circle, and if the two extremities of an equal arch, of an equal lesser circle, be likewise so joined, the two portions of the sphere's surface, thus included, are equal to one another.

For, if the one portion be applied to the other, so that the equal arches of the equal lesser circles shall coincide, the other two boundaries of the figures, namely, the two arches of great circles will also coincide : otherwise, two arches of great circles might pass through two points,

on a sphere's surface, which (Art. 15. and 10.) are not the extremities of a diameter; which (Art. 16.) is absurd. Therefore, the one figure may be placed upon the other, so as wholly to coincide with it: and, consequently, the two figures are equal to one another.

Prop. IV.

(196.) *Theorem.* If two spherical triangles *, having their sides similarly posited, have two sides, of the one, equal to two sides, of the other, each to each, and have, also, the angles contained by those sides equal to one another, they shall be equal.

For it may be shewn, as in E. 4. 1. that the two sides of the one triangle may be made to coincide with the two sides equal to them of the other: therefore (Art. 16.) their bases or third sides, will in that case, coincide; and the two figures, thus wholly coinciding, will be equal to one another.

(197.) Cor. Two spherical triangles, having their sides similarly posited †, are also equal, if the three sides

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* The spherical figures compared in this and the subsequent propositions, are supposed to be on the same sphere, or on equal spheres, unless the contrary be specified.

† Two such triangles may be shewn to be equal, if their sides be not similarly posited; by joining the poles of circles described, about the two triangles, and each of the angular points: from which construction there will result six isosceles spherical triangles, three of which

are

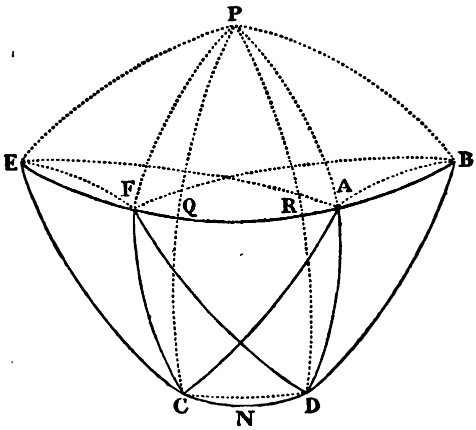
of the one be equal to the three sides of the other, each to each.

For (Art. 86.) the angle contained by any two sides of the one is equal to the angle contained by the two sides equal to them of the other.

Prop. V.

(198.) *Theorem.* The arches of great circles, which join the extremities of two equal arches of equal and parallel lesser circles, in a sphere, toward the same parts, are equal to one another.

Let *AB* and *CD* be two equal arches of the two equal



and parallel circles, *AB* and *CD*; and let *AC* and *BD*

are equal to the other three, each to each: and the equality of the two given triangles will then plainly appear, by taking equals from equals, or else, by adding equals to equals. But as the same thing follows from a subsequent proposition, the proof, which has been pointed out, is not given at length.

be arches of great circles of the sphere, joining A, C and B, D. The arch AC is equal to the arch BD.

Find (Art. 64.) a pole P of the two equal and parallel circles AB, CD; join (Art. 66.) P, C and P, D, and let PC and PD cut EB in Q and R, respectively.

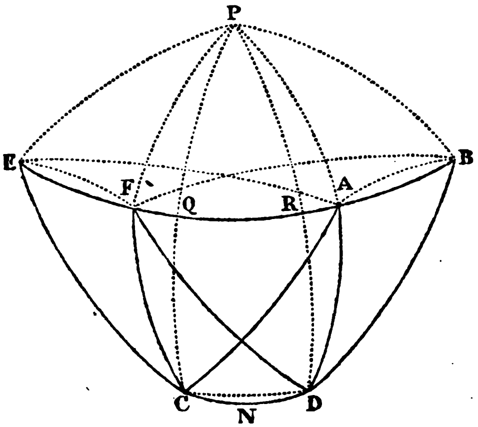
Then, since QR and CD, arches of equal circles, measure (Art. 54.) each of them, the same spherical angle P, they are (Art. 55.) equal to one another. And since QR is equal to CD, and CD, by the hypothesis, is equal to AB, therefore QR is equal to AB; add to each RA, and the arch QRA is equal to RAB: also, (Art. 32.) QC is equal to RD; and (Art. 50.) QC and RD are each of them at right angles to the parallel circles EAB and CD: therefore (Art. 40.) the arch AC is equal to BD.

Prop. VI.

(199.) *Theorem.* If a quadrilateral spherical figure be bounded by arches of great circles, and through two of its opposite sides there pass two equal arches of equal lesser circles of the sphere, the opposite sides and angles of the figure are equal to one another; and it is bisected by the diameter; that is, by the arch of a great circle joining two of its opposite angles.

Let two equal arches of equal lesser circles pass through AB and CD, two opposite sides of the quadrilateral spherical figure $ABDC$, of which AD is a dia-

meter. The opposite sides and angles of the figure are equal to one another; and AD bisects it.



For, (Art. 17.) the side AB is equal to CD; and (Art. 198.) the side AC is equal to BD: since therefore, the three sides of the spherical triangle ACD are severally equal to the three sides of ABD, it is manifest, from Art. 86, that the opposite angles of the quadrilateral spherical figure are equal to one another.

Lastly, since the three sides of the triangle ACD, are, with respect to one another, posited as are the three sides of ABD, and are severally equal to them, therefore, (Art. 196.) the triangle ACD is equal to ABD: that is, the quadrilateral spherical figure $ABDC$ is bisected by its diameter AD.

PROP. VII.

(200.) *Theorem.* Quadrilateral spherical figures upon the same base, and between the same two equal and parallel circles of the sphere, are equal to one another, if

the arches of the parallel circles, subtended by the base and the side opposite to it, in each figure, be equal to one another.

Let *BC** and *ED* be two quadrilateral spherical figures, upon the same base *CD*, and between the same two equal and parallel circles, *EB* and *CND*: also, let the arches *EF* and *AB* be each of them equal to *CND*: the figure *BC* is equal to the figure *ED*.

If the sides *AB*, *EF*, of the figures, opposite to the base *CD*, be terminated in the same point, it is evident, (Art. 199.) that each of them is then the double of the same triangle, and that they are, therefore, equal to one another.

But if the sides *AB*, *EF*, opposite to the common base *CD*, of the figures, be not terminated in the same point, join (Art. 66.) *E*, *A* and *F*, *B* by arches of great circles: and since, by the hypothesis, *EF* is equal to *AB*, if *FA* be added to each of them, the arch *EFA* is equal to *FAB*, and therefore, (Art. 17.) the arches of great circles *EA* and *FB* are equal; and, also, the surface contained by *EA* and *EFA* is equal (Art. 195.) to the surface contained by *FB* and *FAB*: again, (Art. 198.) *DF* is equal to *CE*, and *DB* to *CA*: wherefore, the sides of the triangle *CEA* are equal to the sides of the triangle *DFB*, each to each: and, consequently, (Art. 197.) the triangle *CEA* is equal to *DFB*: from these equals take the equal surfaces *EAFE*, *FBAF*, and there remains the figure *EFAC* equal to *FABD*; and if

* See the figure in Art. 199.

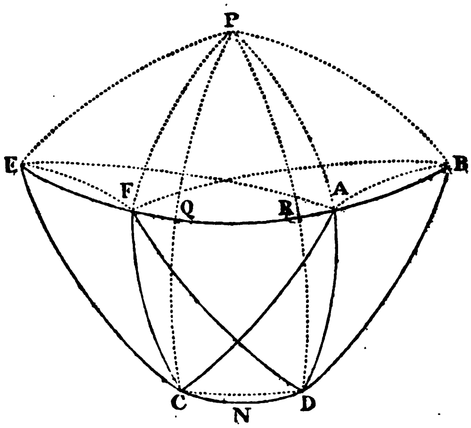
these equals be severally taken from the whole figure *EFABDC*, there remain the quadrilateral figures *BC* and *DE* : therefore, the figure *BC* is equal to the figure *DE*.

(201.) Cor. Hence, it may be shewn, as in E. 36. 1. that quadrilateral spherical figures upon equal bases, and between two equal arches of equal and parallel circles, in a sphere, are equal to one another.

Prop. VIII.

(202.) *Theorem*. Spherical triangles, upon the same base, and between the same two equal and parallel lesser circles of the sphere, are equal to one another.

Let *ADC*, *FCD* be two spherical triangles on the



same sphere, upon the same base *CD*, and between the same parallel and equal circles *EFAB* and *CND*; the angle *ADC* is equal to the triangle *FCD*.

For, find the pole *P* (Art. 64.) of the parallel circles;

join (Art. 66.) P, A and P, F, and P, D and P, C: also, make (Art. 96.) at the point P, the spherical angles FPE and APB, each equal to the angle CPD: wherefore, (Art. 55.) EF and AB are each of them equal to the arch CND: and, if the points E, F, and A, B be joined by arches of great circles, the two quadrilateral figures ED and BC (Art. 200.) are equal to one another: but (Art. 199.) the triangle ADC is the half of BC, and the triangle FCD is the half of ED: therefore, the triangle ADC is equal to FCD.

(203.) Hence, it may be demonstrated, as in E. 38. 1. that spherical triangles upon equal bases, and between the same two equal and parallel circles, of the sphere, are equal to one another.

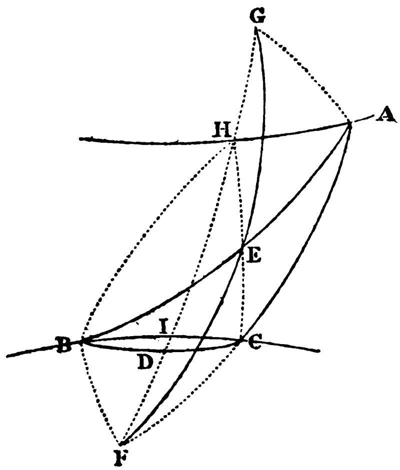
Prop. IX.

(204.) *Problem.* Three points being given, in the surface of a sphere, to describe two equal and parallel circles of the sphere, one of them passing through two of the given points, and the other through the third of the given points: the three given points not being in the circumference of any the same great circle.

Let A, B, C be three given points in a sphere's surface, which are not in the circumference of a great circle: it is required to describe two equal and parallel circles, of the sphere, one of them passing through any two of the points as B, C, and the other passing through the remaining point A.

Join (Art. 66.) A, B and B, C: bisect (Art. 102.) the

arch AB in E, and BC in D: through D, draw (Art. 70.)



the arch of a great circle GDF, at right angles to BC: find, also, (Art. 64.) the polar distance of a great circle in the sphere, and from E as a pole, at that distance, describe a great circle cutting GDF in F: join F, E; then (Art. 36.) FE is a quadrant: let FE and FD produced, meet in G: therefore, (Art. 7.) FEG is a semicircumference, and EG is equal to EF: join G, A and F, B; from G, as a pole, at the distance GA, describe the circle AH; and from F, as a pole, at the distance FB, describe the circle BC: the circle BC passes through C, and is equal and parallel to the circle AH.

For, by the construction, BE, EF are equal to AE, EG, each to each: also, the angles BEF, AEG (Art. 43.) are equal to one another; wherefore, (Art. 97.) the side GA, of the triangle GEA, is equal to the side BF, of the triangle BEF: and therefore, (Art. 38.) the circle AH is equal to the circle BIC: and, because, by the construction,

and Art. 103, the point F is equidistant from B and C, the circle described from F, as a pole, at the distance FB, will pass (Art. 31.) through C.

Lastly, the two equal lesser circles AH and BIC are parallel: for, by the construction, FEG is the semi-circumference of a great circle; wherefore the straight line joining the two poles F, G, is a diameter of the sphere; and (Art. 22.) it is the axis of both the circles AH and BIC: therefore (Art. 25.) they are parallel to one another.

(205.) SCHOLIUM. The problem which is solved in the preceding article, might have been presented in another form; namely, "To find the *locus* of the summits of all the equal spherical triangles, on the same sphere, which are upon the same base, and between the same two equal and parallel circles."

It is manifest, from Art. 202. that the circumference of the circle AH is the locus of the summits of all equal spherical triangles, having the great circle joining B, C for their base, and lying between the same two equal and parallel circles, AH and BIC.

PROP. X.

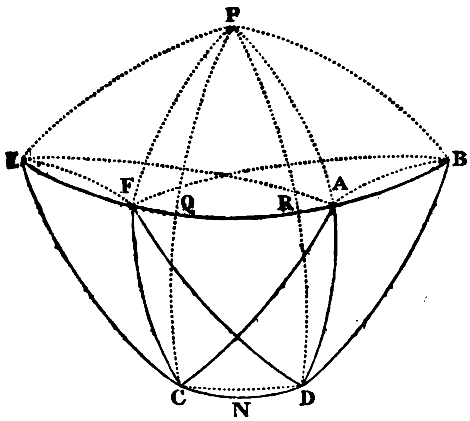
(206.) *Theorem.* Equal spherical triangles upon the same base, and upon the same side of it, or upon equal arches of the same great circle and towards the same parts, are between the same parallel and equal lesser circles of the sphere.

The proposition is proved, by means of Art. 204. 66.

202. and 203. exactly in the same manner as E. 39. and 40. 1.

Prop. XI.

(207.) *Theorem.* If two equal spherical triangles be on the same base, the three angles of the one are, together, equal to the three angles of the other.



Let *ADC*, *FCD* be two equal spherical triangles, on the same base. The three angles of the triangle *ADC* are together equal to the three angles of *FCD*.

Describe (Art. 204.) the two parallel and equal circles *CND* and *BAE*: then (Art. 206.) *AE* passes through the point *F*: let *P* be the pole of *CND*, which was found in describing the parallel circles *CND* and *AE*, make (Art. 92.) the arch *AB* equal to *CD*; and join (Art. 55.) *P*, *C* and *P*, *D* and *P*, *A* and *P*, *B* and *B*, *D*.

Then, (Art. 198.) *AC* is equal to *BD*; and (Art. 31.) *PC* is equal to *PD*, and *PA* to *PB*: therefore, the sides of the triangle *PCA* are equal to the sides of the triangle *PDB*, each to each: and, consequently, (Art.

86.) the angle *PCA* is equal to *PDB*: to these equals add the two angles *ACD*, and *PDC*; and it is manifest, that the two angles *PCD*, *PDC* are, together, equal to the three angles *ACD*, *CDA* and *ADB*: but (demonstration of Art. 199.) the angle *ADB* is equal to the angle *DAC*: wherefore, the three angles of the spherical triangle *ACD* are, together, equal to the two angles *PCD*, *PDC*: and, in the same manner, may the three angles of the spherical triangle *FCD* be shewn to be equal, together, to the same two angles *PCD*, *PDC*: therefore the three angles of the triangle *FCD* are together equal to the three angles of the triangle *ADC*.

(208.) Cor. 1. In the same manner, also, if two equal spherical triangles are upon equal bases, the three angles of the one may be shewn to be equal, together, to three angles of the other: for the isosceles triangle with which the one is thus compared, will have (Art. 31. 56. and 97.) its angles equal, each to each, to the angles of that with which the other triangle is compared.

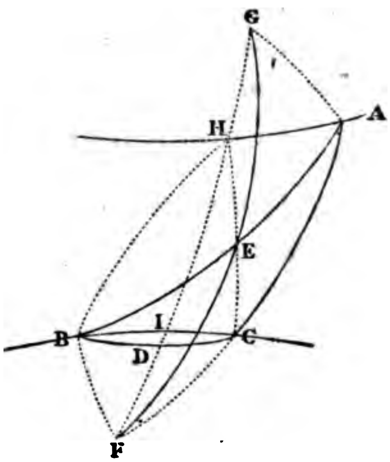
(209.) Cor. 2. It is manifest, from the construction and the demonstration of Art. 207, that a spherical angle may be found, which shall be equal to the semi-aggregate of the three angles of a given spherical triangle.

For, the three angles of the triangle *ADC* were proved to be equal, together, to the two angles at the base of the isosceles spherical triangle *PCD*; which angles (Art. 107.) are equal to one another: wherefore, either of the angles *PCD* or *PDC* is equal to the half of the aggregate of the three angles of the triangle *ADC*.

PROP. XII.

(210.) *Problem.* A scalene spherical triangle being given, to describe upon its base an isosceles spherical triangle, that shall be equal to it.

Let *ABC* be a scalene spherical triangle: it is required to describe on its base *BC*, an isosceles spherical triangle, that shall be equal to it.



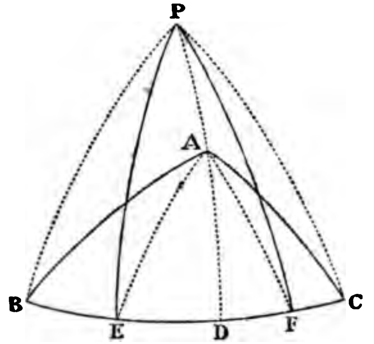
The base, *BC*, having been (Art. 102.) bisected in *D*, and two equal and parallel circles *BIC* and *AH* having been described (Art. 204.) the one passing through *B*, *C* and the other through *A*, draw (Art. 70.) the arch *DH* at right angles to *BC* in *D*; and let *DH* meet *AH* in *H*: join (Art. 66.) *H*, *B* and *H*, *C*: then (Art. 103.) the triangle *HBC* is isosceles; and it is upon the base *BC*; and (Art. 202.) it is equal to the triangle *ABC*.

PROP. XIII.

(211.) *Problem.* A spherical triangle being given

on the surface of a given sphere, to make an isosceles quadrantal triangle, on the sphere, that shall be equal to the given triangle.

Let *ABC* be a given spherical triangle, on the surface of a given sphere: it is required to make an isosceles



quadrantal triangle on that sphere, which shall be equal to the triangle *ABC*.

Find (Art. 64.) the pole *P* of *BC*; join (Art. 66.) *P*, *A*, and produce *PA* to meet *BC*, in *D*. Describe (Art. 204.) two equal and parallel circles, *PB* and *AE*, the one passing through the points *P*, *B*, and the other *AE* through *A*; and let *AE* meet *BC*, in *E*: in like manner, on the other side of *PD*, describe two equal and parallel circles, the one passing through *PC* and the other passing through *A*, and meeting *BC* in *F*. Lastly, join *P*, *E* and *P*, *F*: then, (Art. 36. and 120.) *PEF* is a quadrantal triangle: and it is equal to the given triangle *ABC*.

For (Art. 202. and the construction) the triangle *ABE* is equal to *EPA*, and the triangle *ACF* to *FPA*;

to these equals add the triangle *EAF*, and it is evident that the whole triangle *ABC* is equal to the whole quadrantal triangle *PEF*.

(212.) COR. 1. It is evident, from the construction, that if an isosceles quadrantal triangle were given, a spherical triangle might, by a similar method, be made, which should be equal to the given quadrantal triangle, and have its spherical altitude equal to a given arch.

In the same manner, also, a spherical triangle might be constructed, of a given altitude, and equal to a given spherical segment, contained by the semi-circumferences of two great circles; a quadrantal triangle having first been found, that is equal to the given segment, by taking, for its base, an arch equal to the double of the arch, which joins the bisections of the semi-circumferences.

(213.) COR. 2. The three angles of any given spherical triangle *ABC* are, together, equal to three angles of the quadrantal triangle *PEF*, which is equal to the given triangle.

For, (Art. 207.) the angles of the two triangles *BAE*, *CAF* are together equal to the angles of the two triangles *PAE*, *PAF*; to these equals add the three angles of the triangle *AEF*; and, it is manifest, (Art. 42. and 47.) that the three angles of the triangle *ABC*, together with four right angles, are equal to three angles of the triangle *PEF*, together with four right angles: wherefore, the three angles of the one triangle are, together, equal to the three angles of the other.

Prop. XIV.

(214.) *Theorem.* If two given spherical triangles on the same sphere, or on equal spheres, be equal to one another, the three angles of the one, are, together, equal to the three angles of the other.

For, if two quadrantal triangles be made (Art. 211.) which are equal to the two spherical triangles, each to each, they will, manifestly, be equal to one another: and, since (Art. 121.) the angles at their bases are right angles, if either of them be applied to the other, so that its side, which is a quadrant, coincides with the quadrantal side of that other, it is evident, that the two quadrantal triangles will wholly coincide; otherwise, they must be unequal; which is contrary to the supposition: wherefore, the angles opposite to the bases of the two triangles are equal: and, since their other angles are right angles, the three angles of the one are equal to the three angles of the other: and, consequently, (Art. 213.) the three angles of either of the two given spherical triangles, are, together, equal to the three angles of the other.

(215.) Cor. 1. Conversely, two spherical triangles upon the same sphere, or on equal spheres, are equal, if the three angles of the one be together equal to the three angles of the other.

For, (Art. 213. and 97. and 196.) the isosceles quad-

rantal triangles, which are equal to them, are equal to one another.

(216.) Cor. 2. Hence, if two spherical triangles have the three sides of the one equal to the three sides of the other, or two sides and the included angle in the one, equal to two sides and the included angle, in the other, each to each; or a side and the two angles adjacent, in one of the triangles, equal to a side, and the two adjacent angles, in the other, each to each; in all these cases, the triangles shall be equal, whether their sides be similarly posited or not: because (Art. 86. 97, 98.) the three angles of the one will, in all the cases, be equal to the three angles of the other: and, therefore, (Art. 214.) the two triangles are equal.

Prop. XV.

(217.) *Problem.* To bisect a given spherical surface contained by the semi-circumferences of two great circles, by an arch of a great circle drawn through a given point in either of the semi-circumferences.

Join (Art. 66.) the given point, and a point in the other semi-circumference, so taken (Art. 92.) as that its distance from either of the intersections of the semi-circumferences, shall be equal to the distance of the given point, from the other intersection: and it is manifest, from Art. 87. and 215. that the given surface is thus divided into two equal spherical triangles: that is, it is bisected, by an arch of a great circle, passing through the given point.

Prop. XVI.

(218.) *Theorem.* Isosceles quadrantal triangles having their equal sides quadrants, are to one another as their bases.

The proposition is proved exactly in the same manner as is E. 1. 6.

(219.) Cor. 1. Such isosceles quadrantal triangles are to one another as their vertical angles; that is, as the angles opposite to their respective bases (E. 11. 5.): for the bases (Art. 54.) are to one another as the angles opposite to them.

(220.) Cor. 2. Any such an isosceles quadrantal triangle is, therefore, to the surface of half of the sphere, as its vertical angle is to four right angles.

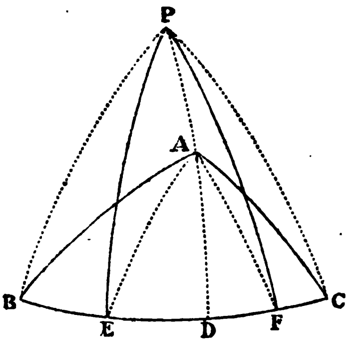
(221.) Cor. 3. A spherical surface, bounded by the semi-circumferences of two great circles, is to the surface of the sphere, as the angle contained by the two semi-circumferences is to four right angles.

Prop. XVII.

(222.) *Theorem.* A spherical triangle is to the half of the sphere's surface, as the excess of the three angles of the triangle, above two right angles, is to four right angles.

Let ABC be a spherical triangle: its surface is to the half of the sphere, as the excess of its three angles A, B, C, is to four right angles.

Make (Art. 211.) the isosceles quadrantal triangle PEF equal to ABC: therefore, (Art. 213.) the three



angles A, B, and C, are, together, equal to the angles PEF, EFP and FPE: but (Art. 121.) the two angles PEF, EFP are right angles: therefore the third angle EPF is equal to the excess of the three angles A, B, C, above two right angles: and (Art. 218.) the quadrantal triangle EPF is to the half of the sphere's surface, as the angle EPF is to four right angles: wherefore, (E. 7. 5.) the given triangle ABC, to which EPF was made equal, is to the half of the sphere's surface, as the excess of the three angles A, B, C, above two right angles, is to four right angles.

(223.) Cor. 1. If two great circles make, with one another, an angle equal to the half of the excess of the three angles of a spherical triangle above two right angles, the

spherical segment, contained by the semi-circumferences of the circles, shall be equal to the spherical triangle: (Art. 221. and 222).

(224.) Cor. 2. The surface of the hemisphere is to any spherical polygon, bounded by arches of great circles, as four right angles are to the excess of all the angles of the polygon, together with four right angles, above twice as many right angles as the polygon has sides: and, the surface of the hemisphere is to its excess above the polygon, as four right angles are to the excess of all the angles of the polygon, above twice as many right angles as the figure has sides.

The Corollary is made evident, by dividing the polygon into as many triangles as it has sides.

Prop. XVIII.

(225.) *Theorem.* If two spherical triangles be on different spheres, and the three angles of the one be, together, equal to the three angles of the other, the two triangles are to one another as the squares of the diameters of their respective spheres.

For (Art. 222.) each of the triangles has to the half of the surface of its sphere, the same ratio that the other triangle has to the half of the surface of its sphere: therefore, (E. 11. and 16. 5.) the triangles are to one another as the surfaces of their respective spheres; that is, (Archim. l. 1. P. 38. and E. 2. 12.) as the squares of the diameters of the spheres.

(226.) Cor. If the three angles of the one, of two spherical triangles, on different spheres, be equal to the three angles of the other, each to each, and if the sides of the triangles be also similarly posited, the triangles are proportional to the squares of the straight lines which are equal to their homologous sides.

For, it was shewn, in the demonstration of Art. 189. that any two homologous sides of two such spherical triangles, are to one another, as the circumferences of great circles, in the two spheres; that is, (Archim. de Dim. Circ. P. 1. E. 2. 12. and 22. 6.) as the diameters of the spheres: wherefore, (E. 22. 6.) the squares of the straight lines, equal to the homologous sides, are to one another, as the squares of the diameters of the spheres; that is, (Art. 224.) as the two spherical triangles.

A

TREATISE

ON

Spherics.

INTRODUCTION

TO

PART II.

ON THE

PRINCIPLES OF PLANE TRIGONOMETRY.

"De Trigonometria libri extant, fateor, non pauci. Sed, ex his, aliqui nimia copia tyronem obruunt; alii difficili brevitate discruciant."——CASWELL.

INTRODUCTION

TO

PART II.

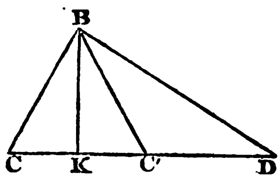
ON THE PRINCIPLES OF PLANE TRIGONOMETRY.

(1.) THE doctrine of Spherical Trigonometry is founded on that of Plane Trigonometry; which may best be learnt from those authors, whose main design is the explanation of its theory, and of its practical uses. All that is intended, therefore, by the following summary, is to recapitulate, and, in as concise a manner as can well be adopted, to demonstrate its principal propositions; with the aim of reviving, rather than of communicating, a knowledge, which is indispensibly necessary in the perusal of the latter part of this Treatise.

(2.) Of the six constituent parts of a plane triangle, namely, the three sides, and the three angles, if there be given, either any two sides, and the included angle, or

the three sides, or any one side together with any two angles, it is evident, from E. 4. 1. E. 8. 1. E. 26. 1. that the triangle is thereby determined: and it might, in reality, be *constructed*, by means of E. 23. 1. and 3. 1. E. 22. 1. E. 23. 1. and 33. 3. according to the case.

But, if two sides and an angle opposite to one of them be given, although the *position* of the third side be thereby given, its *length* is not absolutely determined.



Thus, if *BD* be made equal to one of the given sides, and the angle *BDC* to the given angle, a circle, described from *B* as a center, at a distance equal to the other given side, unless it touch *DC*, will cut it, necessarily, in two points; let these points be *C* and *C'*: Then if *B*, *C* and *B*, *C'* be joined, it is manifest, that the data may belong to either of the triangles *CBD*, *C'BD*. Since, however, a circle cannot cut a straight line in more than two points, there cannot be more than two triangles to which the data may belong.

(3.) In pure Geometry, the *given* quantities are supposed to be actually delineated; and the only instruments which are required, in the subsequent operations, are the rule and compass. But, in mixt mathematics, given quantities are more frequently denoted by *numerical*

values. A given line, it is evident, may be expressed by the number of equal parts, of a specific magnitude, which it contains: and, since (E. 33. 6.) arches of any, the same, circle may be made the measures of any given angles, it is plain, that given angles may be expressed by the number of equal parts, contained in the arches which measure them; the whole circumference being supposed to be divided into some definite number of such equal parts.

The Greeks supposed the whole circumference to be divided into 360 equal parts, called degrees, and marked 360^0; each degree into 60 equal parts called minutes, and marked $60'$, each minute into 60 equal parts called seconds, and marked $60''$; and so on. No geometrical construction can, indeed, be quoted, by means of which this division might actually be effected: but, it is manifestly possible; and it may, therefore, be supposed to be made; which is all that the reader is required to grant.

If the whole circumference had been supposed to be divided into 384^0, each degree into $64'$, each minute into $64''$, and so on, such a division could have really been

* This mode of dividing the circle would have been attended with one considerable practical advantage. It would have enabled the artist to graduate, with greater ease and greater precision, those instruments, by which angular distances are actually measured: for no construction can be more simple, or admit of greater exactness, than that, by which a hexagon is inscribed in a circle, which is the first operation in this case; and all the subdivisions are then effected, merely by the bisection of arches. Such a graduation has, accordingly, in this country been applied to a quadrant.

It has been conjectured, that the Greeks adopted the division of the circle

made, by the help of E. 15. 4. and 9. 1 *. The Grecian mode of division, however, is still generally followed; although the dividing the quadrant into 100 equal parts, instead of 90, which has been partially adopted in France, greatly simplifies and facilitates the numerical calculations of Trigonometry.

If it be granted, that the circumference of a circle can be so divided, it is manifest, that a numerical value may be assigned to any given angle. But, it becomes further necessary, in Practical Geometry, to employ two additional instruments; a scale of equal parts, and a scale of chords; in order first to delineate the data, and afterwards to measure the quantities, determined by the construction of the problem. The first mentioned of these instruments needs no description: and it may be constructed by means of E. 10. 6.

The *Scale of Chords* is a straight line, on which is marked the lengths of the chords of all arches, that contain fewer degrees than 180. If, therefore, a scale of chords be given, the radius of the circle for which it was constructed is given; because (E. 15. 4. Cor.) that radius is equal to the chord of an arch of 60°.

It is evident, also, that, from any given scale of

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circle into 360°, from having observed, that the Sun described the 360th part of his apparent course in a day: and Ptolemy assigns, as a reason for the division of the degree into 60 equal parts, the multitude of the divisors of the number 60.

chords might be constructed, another scale, adapted to a circle of any other given radius, by means of E. 10. 6.

The uses to which such an instrument may be applied are very obvious.

If, from the center of any proposed circular arch, or the summit of any proposed plane rectilineal angle, at a distance equal to the chord of 60^{0}, a circle be described, and the scale of chords be applied to the two points in which its circumference is cut, by two straight lines, produced if necessary, which contain the proposed angle, or which are drawn, from the center, through the extremities of the proposed arch, the number of degrees will be indicated, which measure the proposed arch or angle.

And, conversely, on a given plane, an arch may be described, or an angle may be made, by the help of a scale of chords, which shall have for its measure any given number of degrees.

If, likewise, a scale of chords be constructed for a *great* circle, in a given sphere, any proposed arch of such a circle may thereby, very readily, be measured. For, the distance between the extremities of the arch may easily be taken, by the opening of a spherical compass, and may thus be transferred to the scale of chords. And, by a converse operation, there may be cut off, from the circumference of a given great circle of the sphere, an arch of any proposed number of degrees.

It is plain also, from what has been premised, and from Part I. Art. 61. that the number of degrees in a given arch of *any* circle, of a given sphere, may be found, by means of a spherical compass, and *any* given scale of chords: and, also, that with the same instruments, an arch may be cut off, from the circumference of *any* given circle of the sphere, which shall have for its measure any proposed number of degrees.

(4.) If there be given three of the parts, of which one is a side, of a plane triangle, to find the remaining parts, it has already appeared (2, 3 *.) how far the problem is possible, and how far it may be solved by *pure Geometry*, or by what are termed *Graphical Operations*. It is the business of *Plane Trigonometry*, properly so called, to solve this problem so far as it is possible, by the method of *computation* alone.

The advantage peculiar to the method of computation is, that it may be carried to any required degree of exactness: the precision of geometrical construction, is on the other hand necessarily limited, by the imperfection as well of the instruments employed, as of the senses of the operator.

(5.) Def. The *Complement of an arch* is the differ-

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* The numbers 2 and 3 refer to the Articles of this Introduction: the same may be said of similar references, where the contrary is not specified, throughout the Introduction.

ence between that arch and a quadrant: and the *Supplement of an arch* is the difference between that arch and two quadrants.

The *Complement of an angle* is, likewise, the difference between that angle and a right angle: and the *Supplement of an angle* is the difference between that angle and two right angles.

(6.) Def. The *Sine* of an arch is a perpendicular let fall, from either of its extremities, upon the diameter passing through the other extremity: and its *versed sine*, is the part of that diameter which is included between the sine and the extremity of the arch.

(7.) Cor. 1. The sine of a quadrant is the radius of the circle: and the sine of half a quadrant (E. 32. 1. 47. 1.) is equal to the quotient of the radius divided by $\sqrt{2}$: that is, according to the division of the circle, described in Art. 3. if R be put for the radius, $\sin 90^\circ = R$; and $\sin 45^\circ = \frac{R}{\sqrt{2}}$.

(8.) Cor. 2. The sine of any arch is equal to half the chord of the double of that arch, and conversely, (E. 3. 3. and 30. 3.)

(9.) Cor. 3. Hence, since (E. 15. 4. Cor. and E. 32. 1.) the chord of an arch of 60° is equal to the radius, the sine of an arch of 30° is equal to the half of the radius.

(10.) Cor. 4. The versed sine of any arch is a third proportional (E. 8. 6.) to the diameter, and the chord of that arch, or (Art. 7.) to the diameter and the double of the sine of the half of that arch.

(11.) Def. The *Tangent* of an arch is a straight line, drawn from either extremity of the arch, at right angles to the diameter passing through that extremity, and terminated by the diameter produced, which passes through the other extremity: and the part of that produced diameter, which lies between the center and the tangent, is called the *Secant* of the arch.

(12.) Cor. The tangent of the half of a quadrant is (E. 32. 1. and 6. 1.) equal to the radius of the circle: if, therefore, A^0 be any circular arch, and R be the radius of its circle, then,

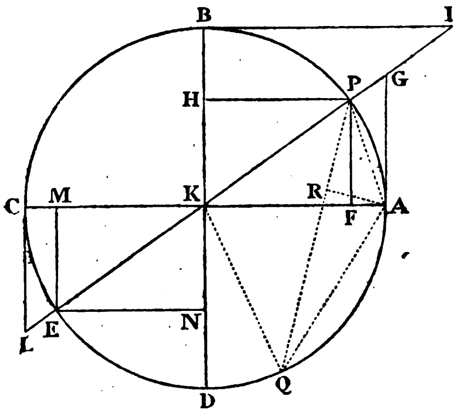
$$\tan 45^0 = R$$

$$\text{also } \sec A = \sqrt{(R^2 + \tan^2 A)}. \quad \text{(E. 47. 1.)}$$

(13.) The *cosine, co-tangent,* and *co-secant* of an arch, are, respectively, the sine, the tangent, and the secant of its complement.

Thus, if AP be any arch, less than a quadrant, of the circle $ABCD$, having K for its center, let CA and EP be the two diameters passing through A and P; let the diameter BD, and also the straight lines PF, and EM, be drawn perpendicular to CA; and let EP produced, both

ways, meet the tangents *AG, BI, CL*, in *G, I* and *L*,



respectively; also let *PH* and *EN* be drawn perpendicular to *BD*.

Then is *BP* the complement, and *CBP* the supplement of *AP*.

Also, *BCE* is the complement, and *CE* the supplement of the arch *ABCE*.

PF is the sine of *AP*, and of *CBP*; *PH* is the cosine of *AP*, and of *CBP*.

AG is the tangent, and *KG* the secant of *AB*, and of *ECBA*.

CL is the tangent, and *KL* the secant, of *CBP*.

BI is the co-tangent, and *KI* the co-secant, of *AP*, and of *CBP*.

EM is the sine, and *EN* the cosine, of *ABCE*, and of *CE*.

FA is the versed sine of *AP*.

(14.) Cor. 1. The sine, cosine, tangent and secant

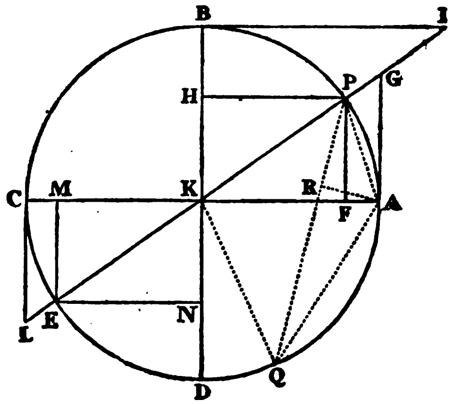
of any arch, are respectively equal in magnitude, to the sine, cosine, tangent and secant of the supplement of that arch.

(15.) COR. 2. That part of a diameter, passing through the extremity of an arch, which lies between its sine, and the center of the circle is (E. 34. 1.) equal to the cosine of the arch, and may always be put for the cosine. If, therefore, A be any arch, and R the radius of the circle, $\sin^2 A + \cos^2 A = R^2$.

(16.) COR. 3. Since (E. 33. 6.) any two straight lines drawn from the common center of two concentric circles cut off from them arches which have the same ratio, each to the whole circumference, it is evident, that the sine of an arch of any number of degrees in the one, will be to the sine of an arch of the same number of degrees in the other as the radius of the former, to the radius of the latter (E. 4. 6.): and so of the chords and of the rest of the trigonometrical functions which have been defined, of two arches, of the same number of degrees. So that if the magnitude be given of any such function of an arch of A^0 in a circle of any given radius, the same function may be found, of an arch of A^0 in a circle of any other given radius.

(17.) COR. 4. The right-angled triangles KPF, KGA, KPH, KIB are all (E. 29. 1.) equiangular with one another: therefore, (E. 4. 6.)

$$KF : FP :: KA : AG$$
$$FP : PK : AG : GK$$
$$AG : AK : KB : BI$$
$$FP : PK :: BK : KI.$$



Hence, if unity denote the radius, and A any arch of the circle,

$$\tan A = \frac{\sin A}{\cos A};$$

$$\sec A = \frac{\tan A}{\sin A} = \frac{1}{\cos A}; \therefore (12.) \cos A = \frac{1}{\sqrt{(1 + \tan^2 A)}};$$

$$\cot A = \frac{1}{\tan A} = \frac{\cos A}{\sin A}$$

$$\tan A \cot A = 1; \text{ and, therefore, } \sin A \operatorname{cosec} A = 1.$$

(18.) Cor. 5. If the sine be given of an arch of a circle, of any given radius, the sine, cosine, tangent, co-tangent, secant and co-secant may be found, of an arch, of the same number of degrees, in a circle of any other given radius (15. 16. 17).

(19.) Cor. 6. It is manifest, from Art. 6. 11. 13, that the greatest value of a sine, or a cosine, is the radius of the circle; but that a sine, or a cosine, may be found, that is less than any given finite straight line: and that the values of a tangent, or a co-tangent, a secant, or co-secant, may be either indefinitely small, or indefinitely great.

Also, since the sines of all arches, having a common extremity, that are less than two quadrants, lie on the one side of the diameter, passing through that extremity, and the sines of all of them that exceed two quadrants, lie on the contrary side, if the sines of the former be accounted *positive*, the sines of the latter must be taken as *negative*. The sines therefore, are positive, through the two first of the four quadrants, and negative through the two last: so that the sines of two arches that are supplements, each of the other, are both equal (14.) in magnitude, and also have the same sign.

Again, the cosines of all the arches which are less than one quadrant, and greater than three quadrants, fall (13.) on the same side of the diameter, to which they are drawn perpendicular; and the cosines of all the arches which are greater than one quadrant, and less than three quadrants, fall on the contrary side of that diameter: so that the cosines are positive in the first, and in the fourth, quadrants, and negative in the second and in the third. The signs of the remaining trigonometrical functions of an arch, according to its magnitude, are determined by the equations investigated in Art. 17.

(20.) Def. The sine, cosine, tangent, co-tangent, secant, co-secant and versed sine of an *arch*, are also called the sine, cosine, tangent, co-tangent, secant, co-secant, and versed sine respectively, of any *angle*, of which that arch is the measure *. So that the sine of a right angle is (7.) the radius of the circle: and the sine of either of the two acute angles of a right-angled plane triangle is (Art. 13. and E. 32. 1.) the cosine of the other.

(21.) The sides of a plane triangle are proportional to the sines of the angles opposite to them.

For, if a circle be described (E. 5. 4.) about any plane triangle, the sides (E. 21. 3. and Art. 8. 20.) are the doubles of the sines of the opposite angles, in that circle; and (Art. 16.) these sines are proportional to the sines of the same angles in any other circle: wherefore, (E. 11. 5.) the sides of a triangle are to one another as the sines of the opposite angles.

(22.) Cor. 1. The sines of all angles, in a circle of any given radius, being supposed to be known, or the ratios of those sines to the radius being known, if there be given a side and the opposite angle, together with any other part of a plane triangle, the three remaining parts may be found, by Art. 21.

* The sine of an *angle* is, by some authors, defined to be a fraction, of which the numerator has to the denominator the same ratio as the sine of an arch, taken in any circle, so as to measure the angle, has to the radius of the circle. The value of the sine of any given angle thus becomes a constant value. Corresponding definitions are, also, given of the remaining trigonometrical functions of an angle.

(23.) Cor. 2. If A denote the right angle, and S the hypotenuse of any proposed right-angled plane triangle, A', A'', the two acute angles, S', S'', the two sides respectively opposite to them, and if unity be the radius of the circle, by the arches of which the angles of the triangle are measured, then (Art. 21. and 20.)

$$S : S' :: 1 : \sin A' \text{ or } \cos A''$$
$$S : S'' :: 1 : \sin A'' \text{ or } \cos A'$$
$$S' : S'' :: \sin A' : \sin A'' \text{ or } \cos A';$$

$$\therefore \ (1.) \ \frac{S'}{S} = \sin A' = \cos A''$$

$$(2.) \ \frac{S'}{S''} = \frac{\sin A'}{\cos A'} = \tan A' \ (\text{Art. } 17.)$$

And it is to be remarked, that the two last terms, in each of the above proportions, are the sides of triangles, which are similar to the proposed triangle, and which may be called *Functional Triangles.*

Hence, the circle being supposed to be graduated, in the manner before described, if the lengths of the sines and the tangents, of all angles that are less than right angles, be known, or if the proportions, which these lines severally bear to the radius, be known, then it is plain, that whenever two sides of a proposed right-angled plane triangle are given, a side of one of the functional triangles may be found, which will indicate one of the angles of the proposed triangle: and that, if a side and one of the acute angles be given, the other two sides may be found; because, upon the supposition made, a side of the functional triangle is known, if one of the acute angles be given. So that in

either case, three out of the four terms of one of the two proportions, above investigated, will be known: and, therefore, if any two parts, of which one is a side, of a right-angled plane triangle, be given, besides the right angle, the remaining parts may be found. It is evident also, that if two sides of such a triangle be given, the third side may be found, from the equation, $S^2 = (S')^2 + (S'')^2$ (E. 47. 1.)

(24.) Cor. 3. If A and A' denote the angles at the base of any plane triangle, S and S' the sides opposite to them, D and D' the segments into which the base is divided, and V and V' the segments into which the vertical angle is divided, by a perpendicular (L) drawn from it to the base, D and V being the segments adjacent to the side S', then (E. 47. 1.)

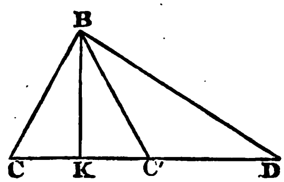
$$L^2 = S^2 - D'^2 = (S')^2 - D^2;$$
$$\therefore S^2 \sim S'^2 = D'^2 \sim D^2$$
$$\therefore (S \sim S')(S + S') = (D' \sim D).(D + D') = (D' \pm D).S'';$$
$$\therefore \frac{S''}{S + S'} = \frac{S \sim S'}{D' \pm D}.$$

If three parts, therefore, of which one is a side, of an oblique-angled plane triangle be given, the remaining parts may be found, when the three sides are given, by the equation last deduced, and by Art. 23; and in all the other cases by Art. 23; the proposed triangle having been first divided into two right-angled triangles, by a perpendicular let fall, from some one of its angles, upon the opposite side; the same supposition also being made, with respect to the graduated circle as in Art. 23: With the ex-

M 2

ception, therefore, of what relates to the computation* of the proportionate magnitudes of the sines and the tangents of given angles, this and the preceding article may be said to comprehend the whole business of Plane Trigonometry, as it is defined in Art. 4. There are, however, more expeditious methods of solving plane triangles; and these remain to be investigated.

Let BK, drawn from the angle B, perpendicular to



CD, fall within the triangle CBD, and without the triangle $C'BD$. Then (E. 12. and 13. 2.)

$$BD^2 = BC^2 + CD^2 - 2DC.CK$$
$$= BC'^2 + C'D^2 + 2.DC'.C'K;$$

that is, (Art. 21. 13. 19.) $S^2 = (S')^2 + (S'')^2 - 2S''S' \cos A$;

$$\therefore \cos A = \frac{(S')^2 + (S'')^2 - S^2}{2\,S'\,S''}.$$

(26.) Cor. $\text{Sin}^2 A = 1 - \cos^2 A = 1 - \frac{((S')^2 + (S'')^2 - S^2)^2}{4\,(S')^2\,(S'')^2}$

$$= \frac{4(S')^2(S'')^2 - ((S')^2 + (S'')^2 - S^2)^2}{4\,S'^2\,S''^2} =$$

$$= \frac{(S+S'+S'').(S-S'+S'').(S+S'-S'').(S'+S''-S)}{4\,(S')^2\,(S'')^2}.$$

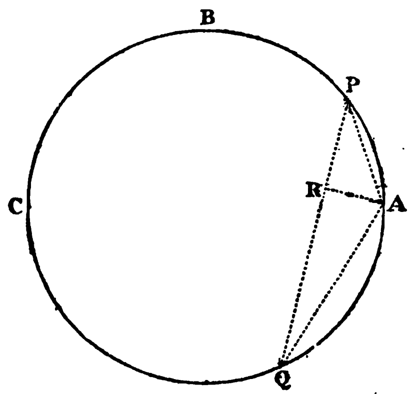
And, by means of this equation, the sine of any angle

* The practicability of making this computation will be shewn in the last Article of this Introduction.

of a plane triangle may readily be found, when the three sides are given.

(27.) To express, in terms of the sines and the cosines of two given arches (a) and (b), the sines of their sum, and of their difference.

Let AQ, in the circle $ABCQ$, be the *double* of (a), and AP, the *double* of (b): join A, P, and A, Q, and



P, Q; and draw AR perpendicular to PQ. Then (Art. 8.) $AQ = 2 \sin a$; $AP = 2 \sin b$; $PQ = 2 \sin(a+b)$: but

$$PQ = QR + RP$$
$$= AQ \cos AQP + AP \cos APQ \text{ (23.)}$$

that is, $2 \sin(a+b) = 2 \sin a \cos b + 2 \sin b \cos a$ (20. and E. 20. 3.)

$$\therefore \sin(a+b) = \sin a \cos b + \sin b \cos a.$$

And, in the same manner, if PQ be taken the double of the greater, and AQ the double of the less, of two

given arches, a and b, by letting fall a perpendicular from Q upon PA produced, it may be shewn that,

$$\sin(a-b) = \sin a \cos b - \sin b \cos a.$$

It is manifest, however, that if $a=b$, $\sin(a-b) = 0$.

(28.) Cor. 1. Let c be the complement of the arch b. Then,

$$\sin(a-c) = \sin(a+b-90^0)$$
$$= -\cos(a+b) \qquad (19.)$$

But,

$$\sin(a-c) = \sin a \cos c - \sin c \cos a \qquad (27.)$$
$$= \sin a \sin b - \cos a \cos b \qquad (13.)$$

$$\therefore -\cos(a+b) = \sin a \sin b - \cos a \cos b,$$

and

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

And, in a similar manner, it may be shewn, that

$$\cos(a-b) = \cos a \cos b + \sin a \sin b.$$

It is evident, however, (Art. 13. 7.) that if $a=b$, $\cos(a-b) = 1$.

(29.) Cor. 2. Since (Art. 17.) $\tan(a \pm b) = \dfrac{\sin(a \pm b)}{\cos(a \pm b)}$.

$$\tan(a \pm b) = \frac{\sin a \cos b \pm \sin b \cos a}{\cos a \cos b \mp \sin a \cos b}$$

$$= \frac{\dfrac{\sin a}{\cos a} \pm \dfrac{\sin b}{\cos b}}{1 \mp \dfrac{\sin a}{\cos a} \cdot \dfrac{\sin b}{\cos b}}$$

$$= \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}; \qquad (17.)$$

$$\therefore \cot(a \pm b) = \frac{1 \mp \tan a \tan b}{\tan a \pm \tan b} \qquad (17.)$$

(30.) Cor. 3. From *adding*, and *subtracting*, the equations investigated in Art. 27. 28. there results,

$$\sin a \cos b = \tfrac{1}{2} \sin (a+b) + \tfrac{1}{2} \sin (a-b)$$
$$\cos a \sin b = \tfrac{1}{2} \sin (a+b) - \tfrac{1}{2} \sin (a-b)$$
$$\cos a \cos b = \tfrac{1}{2} \cos (a-b) + \tfrac{1}{2} \cos (a+b)$$
$$\sin a \sin b = \tfrac{1}{2} \cos (a-b) - \tfrac{1}{2} \cos (a+b).$$

Again, if b be successively made equal to $\tfrac{1}{2} a$, a, $2a$, $3a$, &c. in the equations of Art. 27. 28. 29. then,

$$\text{first,} \quad \sin \tfrac{1}{2} a = \sqrt{\frac{1 - \cos a}{2}}$$

$$\cos \tfrac{1}{2} a = \sqrt{\frac{1 + \cos a}{2}}$$

$$\tan \tfrac{1}{2} a = \sqrt{\frac{1 - \cos a}{1 + \cos a}} \quad (17.)$$

$$\cot \tfrac{1}{2} a = \frac{1 + \cos a}{\sin a}$$

$$\cos a = \frac{1 - \tan^2 \tfrac{1}{2} a}{1 + \tan^2 \tfrac{1}{2} a}.$$

And, it is evident, that by means of the three first of the above forms, the sine, the cosine, the tangent, and the co-tangent of $\frac{a}{2^n}$ may be found.

Secondly,

$$\sin 2a = 2 \sin a \cos a$$
$$\sin 3a = \sin a \cos 2a + \sin 2a \cos. a$$
$$\&c. = \&c.$$
$$\cos 2a = \cos^2 a - \sin^2 a$$

$$\cos 3\,a = \cos a \cos 2\,a - \sin a \sin 2\,a$$
$$\&c. = \&c.$$
$$\tan 2\,a = \frac{2 \tan a}{1 - \tan^2 a}$$
$$\&c. = \&c.$$
$$\cot 2\,a = \frac{\cot a - \tan a}{2}.$$

Whence may be found the sine, the cosine, the tangent, and the co-tangent of $2^n . a$.

Lastly, if in the equations of Art. 27. 28. b be successively made equal to $2\,b$, $3\,b$, &c. it will appear that the sines, and the cosines, of arches, which are in arithmetic proportion, form a recurring series; so that any term of it may be deduced from the two next antecedent terms, by multiplying the nearer of the two by the cosine of the common difference of the arches, and the other term by -1.

(31.) Cor. 4. If the values of $\sin(a \pm b)$ and $\cos(a \mp b)$, in Art. 27. 28. be *multiplied* together, it follows from Art. 15. 30. that

$$\sin a + \sin b = 2 . \sin \tfrac{1}{2}(a+b) \cos \tfrac{1}{2}(a-b).$$
$$\sin a - \sin b = 2 . \cos \tfrac{1}{2}(a+b) \sin \tfrac{1}{2}(a-b).$$
$$\cos a + \cos b = 2 . \cos . \tfrac{1}{2}(a+b) \cos \tfrac{1}{2}(a-b).$$
$$\cos b - \cos a = 2 . \sin \tfrac{1}{2}(a+b) \sin \tfrac{1}{2}(a-b).$$

And, if the first of these equations be divided by the three last, and the second of them by the third and fourth, there results (17.)

$$\frac{\sin a + \sin b}{\sin a - \sin b} = \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)}$$

$$\frac{\sin a + \sin b}{\cos a + \cos b} = \tan \tfrac{1}{2}(a+b)$$

$$\frac{\sin a + \sin b}{\cos b - \cos a} = \cot \tfrac{1}{2}(a-b)$$

$$\frac{\sin a - \sin b}{\cos a + \cos b} = \tan \tfrac{1}{2}(a-b)$$

$$\frac{\sin a - \sin b}{\cos b - \cos a} = \cot \tfrac{1}{2}(a+b)$$

$$\frac{\cos a + \cos b}{\cos b - \cos a} = \frac{\cot \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)}.$$

(32.) COR. 5. $\operatorname{Tan} a \pm \tan b = \dfrac{\sin a}{\cos a} \pm \dfrac{\sin b}{\cos b}$ (17.)

$$= \frac{\sin a \cos \pm \sin b \cos a}{\cos a \cos b}$$

$$= \frac{\sin (a \pm b)}{\cos a \cos b} \quad (27.)$$

And, by a similar process, it may be shewn that

$$\cot b \pm \cot a = \frac{\sin (a \pm b)}{\sin a \sin b}.$$

(33.) COR. 6. If A, A', A'' be the three angles, and S, S', S'' the opposite sides of a plane triangle, then

(Art. 23.) $\dfrac{S + S'}{S \sim S'} = \dfrac{\sin A + \sin A'}{\sin A \sim \sin A'}$

$$= \frac{\tan \frac{1}{2}(A + A')}{\tan \frac{1}{2}(A \sim A')} \quad (31.)$$

$$= \frac{\cot. \frac{1}{2} A''}{\tan \frac{1}{2}(A \sim A')} \quad (13.14.\ \text{E}.32.\ 1.)$$

Whenever, therefore, any two sides, S and S', and the included angle A'' of a plane triangle are given, the co-tangent of the given angle being supposed to be known, the tangent of the difference of the angles at the base, and the difference itself, may be found: and by combining this difference with the sum of the same angles, which is known (E. 32. 1.) because the third angle is given, the angles themselves will be found, and the triangle may be solved by the preceding articles.

(34.) The surface (T) of a plane triangle, is equal (E. 41. 1.) to half of the rectangle contained by its base S and the perpendicular L, let fall upon S, from the opposite angle A;

$$\begin{aligned} \therefore T &= \tfrac{1}{2} L \,.\, S, \\ &= \tfrac{1}{2} S' \,.\, S'' \,.\, \sin A \,.\, (21.\ 23.) \\ &= \sqrt{P \,.\, (P-S) \,.\, (P-S') \,.\, (P-S'')} \quad (26.) \end{aligned}$$

P being put for the semi-sum of the three sides S, S', S'' of the triangle.

(35.) It has been supposed, in the preceding articles, that the length of the trigonometrical functions of a given arch or angle, or the proportions which they severally bear to the radius of the circle, in which they are drawn, can be found. The practicability of this computation remains to be shewn.

It appears, from Art. 18, that if the sine of any proposed arch or angle can be found, in any given circle, all the other functions will thence become known. The

problem, therefore, is reduced to the computation of the sine of an arch, of a given number of degrees, in a circle of which the given radius is denoted by unity.

Again, it follows, from Art. 30, that if the sine of $1''$, of $1'$, or of 1^0, can be found, so likewise can the sine of any number of seconds, minutes, or degrees be found. If, therefore, a method can be shewn by which the sine of $1''$ can be calculated, the problem may be considered as solved.

Now, since (Art. 17.) $\frac{\sin A}{\tan A} = \cos A$, and since, if A be supposed to be indefinitely diminished, $\cos A$ will (Art. 19.) approximate, indefinitely, to *unity*, the value of the radius; it is plain, that the sine and the tangent of a very small arch will be nearly equal: and, since any arch is manifestly greater than its sine, and less than its tangent, the value of a very small arch will not greatly differ from that of its sine: so that very small arches may be considered as proportional to their sines.

But (Art. 9.) $\sin 30^0$ is given, being $\frac{1}{2}$, when the radius is unity; and $\frac{30^0}{2^{17}} = 48''', 11^{\text{iv}}$: wherefore, (Art. 30.) the sine of $48''', 11^{\text{iv}}$ may be considered as determined: let it be denoted by t. Then, $48''', 11^{\text{iv}} : 1'' :: t : \sin 1''$; $\therefore \sin 1'' = \frac{1''}{48''', 11^{\text{iv}}} \times t = \frac{3600}{2891} . t$. Whence, the sine of $1''$ having been thus computed, the sines, and therefore, also, the other trigonometrical functions, may be

found, of an arch of any given number of degrees, minutes, and seconds. Accordingly, sin $2'' = 2 \sin 1''$; $\sin 3'' = 3 \sin 1''$, and so on.

And, in order to find the sines of greater arches, recourse must be had to the equation $\sin 2A = 2 \sin A \cos A$.

In practice, it is usual to compute the sines, cosines and tangents of angles, by means of series, and most commonly by means of those investigated by Euler, which express the values of $\sin (mA)$, $\cos (mA)$, $\tan (mA)$. And when the tangents have been computed, the secants are found, without any other operation, than that of subtraction; for $\sec A = \cot \frac{1}{2} (90^0 - A) - \tan A$. The versed sines are, also, readily found, from the equation ver $\sin A = 1 - \cos. A$.

(36.) It may be remarked, also, that if the sines, cosines, tangents and co-tangents have been computed for all arches from 0^0 to 45^0, amongst them will be found the sines, cosines, tangents and co-tangents of all arches whatever; because, the sine and tangent of any arch *greater* than $45°$ are the cosine and co-tangent respectively, of an arch that is *less* than $45°$.

Further, if the sines and cosines of all arches, from 0^0 to $30°$ be computed, the sines and cosines of all arches whatever may thence be found, merely by subtraction.

For (Art. 30.) $\sin a \cos b = \frac{1}{2}\sin(a+b) + \frac{1}{2}\sin(a-b)$

$$\sin a \sin b = \tfrac{1}{2}\cos(a-b) - \tfrac{1}{2}\cos(a+b).$$

And these equations, if $a = 30^{0}$, and therefore, $\sin a = \frac{1}{2}$, become

$$\cos b = \sin(30^\circ + b) + \sin(30^\circ - b)$$
$$\sin b = \cos(30^\circ - b) - \cos(30^\circ + b)$$
$$\therefore \sin(30^{0} + b) = \cos b - \sin(30^\circ - b)$$
$$\cos(30^\circ + b) = \cos(30^\circ - b) - \sin b.$$

So that if for b be substituted, succcssively, all the arches between 0° and 30°, there will be found the sines and cosines of all arches between 30° and 60°: and amongst the sines and cosines of all the arches between 0° and 45° will be found the sines and cosines of all arches whatever.

Lastly, if the tangents and co-tangents be computed, of all arches between 0° and 30°, the tangents and co-tangents of all arches whatever, that are measured by an *even* number of seconds may be found, without any other arithmetical operation, than a subtraction, and a division by 2. For (Art. 30.) $\cot. 2a = \frac{1}{2}(\cot. a - \tan. a)$. If, therefore, $a = 30^{0} - b$, and $2a = 60^\circ - 2b$, and $\cot(60^\circ - 2b)$

$$= \tan(30^\circ + 2b) = \frac{\cot(30^\circ - b) - \tan(30^\circ - b)}{2}.$$

Whence, the tangents and co-tangents of all arches between 30° and 60° may be found from the tangents and co-tangents of arches that are less than 30°; and thus the tangents and co-tangents of all arches whatever

will be known, when those of all arches from 0° to 45° have been found.

But it is manifest, that a and b in the two forms,

$$\cot 2a = \tfrac{1}{2} (\cot a - \tan a)$$

$$\tan (30^\circ + 2b) = \frac{\cot (30^\circ - b) - \tan (30^\circ - b)}{2}$$

denote *whole* numbers of seconds; and therefore, the arch $(30^\circ + 2b)$ of which the tangent is thus found, must always consist of an *even* number of seconds.

The considerations, however, which have been suggested, in this last article, indicate a very great abridgement of labour in the actual computation of the trigonometrical functions of circular arches.

A

TREATISE

ON

Spherics.

PART II.

SPHERICAL TRIGONOMETRY.

PART II.

THE ELEMENTS OF

Spherical Trigonometry.

SECTION I.

ON THE INVESTIGATION OF SUCH GENERAL PROPOSITIONS AS ARE APPLICABLE TO THE PURPOSES OF SPHERICAL TRIGONOMETRY.

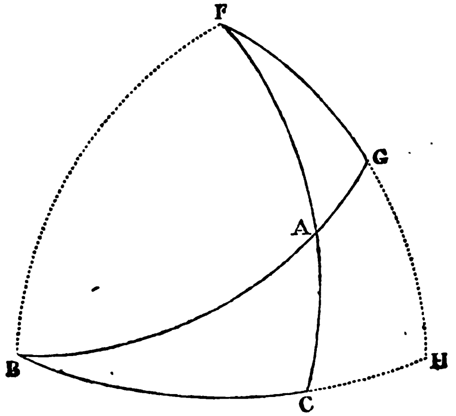
DEFINITION.

(227.) *SPHERICAL Trigonometry* is that part of Spherics, in which are investigated the relations existing between the sides, the angles, and the surface of a spherical triangle, with a view to the resolution of the following general problem: "Of the six first quantities, namely, those made up of the three sides, and the three angles, of a proposed spherical triangle, any three being given, to determine the rest *."

* That this problem involves no impossibility, is very evident, from what has been already proved, in treating of Spherical Geometry.

(228.) DEF. If a given right-angled spherical triangle have two angles, that are not right angles, and from the summit of either of them, as a pole, a great circle be described, cutting the opposite side and the hypotenuse produced, if necessary, the triangle contained by the segments of the circumference so described, and of the two sides which it cuts, is called the *Complemental Triangle* of the given triangle.

Thus, let the angle C, and no other angle, of the spherical triangle ABC, be a right angle; and from either



of the oblique angles, B, as a pole, let there be described the great circle FG, cutting the opposite side AC, and the hypotenuse BA, produced, if necessary *, in F and G: the triangle FAG is one of the complemental triangles of

* In the figure, the two sides of the given spherical triangle are supposed to be of the *same species*, and to be, each of them, less than a quadrant: there are, therefore, two other cases, which might be illustrated by separate figures: but, as the reasoning is general and very easy to be understood, it seems unnecessary to exhibit more than one of the cases.

ABC; and by a similar construction, if a great circle be described from *A*, as a pole, the other complemental triangle will be found.

(229.) Cor. Let the circle *FG* meet the side *BC*, produced, if necessary, in *H*.

Then (Art. 36.) *BG* and *BH* are quadrants; so that *AG* is the complement of *BA*, and *CH* of *BC*. Again, since the angle *C* is a right angle, the pole of *BCH* is (Art. 50.) in *CF*; and, by the construction, *BF* is a quadrant; wherefore, (Art. 36. 50.) *F* is the pole of *BCH*; and, consequently, *FA* is the complement of *AC*: also (Art. 54.) *CH*, the complement of *BC*, measures the angle *AFG*; and *GH*, the complement of *FG*, measures the angle *ABC*; and *FG*, likewise, measures the angle *FBG*, the complement of *ABC*.

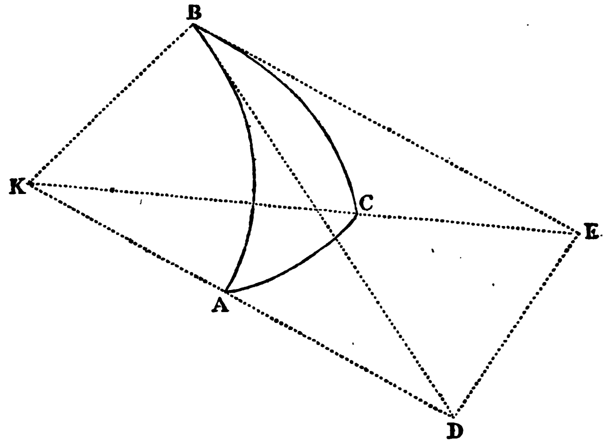
It is manifest, also, that similar conclusions may be drawn, from the other complemental triangle.

Prop. I.

(230.) *Theorem.* The cosine of any one of the sides, of a spherical triangle, is equal to the product of the cosines of the other two sides, together with the continued product of the sines of those two sides, and the cosine of the angle contained by them; unity being put, as well for the tabular radius, as for the radius of the sphere.

Let *AC* be any one of the sides of the spherical tri-

angle ABC; then, unity being put for the tabular radius,



and also for the radius of the sphere,

$\cos AC = \cos AB \cos BC + \sin AB \sin BC \cos \angle ABC.$

For, let K be the sphere's center, and KA, KB, KC, each of them a radius of the sphere; also, let BD and BE be two straight lines, which touch the arches AB and BC, in their common point B, and meet KA and KC, produced, in D and E: and let the two points D, E be supposed to be joined, by the straight line DE.

Wherefore, (Introd. 25.)

$$KD^2 + KE^2 - 2KD \cdot KE \cdot \cos \angle DKE = DE^2$$

also, $$BD^2 + BE^2 - 2BD \cdot BE \cdot \cos \angle DBE = DE^2.$$

Whence, by subtraction, if unity be put for $(KD^2 - BD^2)$, and for $(KE^2 - BE^2)$, which are each of them (E. 18. 3. and 47. 1.) equal to the square of the radius, that is, to unity, if, also, the spherical angle ABC be put

(Art. 41.) for the plane angle DBE, and the arch AC (E. 33. 6.) for the angle DKE,

$$1 + 1 + 2BD.BE.\cos \angle ABC - 2KD.KE.\cos AC = 0,$$

that is, $2 + 2BD.BE.\cos \angle ABC - 2.KD.KE.\cos AC = 0$;

$$\therefore 1 + BD.BE.\cos \angle ABC - KD.KE.\cos AC = 0.$$

But, (Introd. 17.) $BD = \tan AB = \dfrac{\sin AB}{\cos AB}$;

and $BE = \tan BC = \dfrac{\sin BC}{\cos BC}$;

also (Introd. 17.)

$$KD = \sec AB = \frac{1}{\cos AB};$$

and $KE = \sec BC = \dfrac{1}{\cos BC}$:

and, if these values be substituted in the equation last found, it will become, when cleared of fractions,

$$\cos AB \cos BC + \sin AB \sin BC \cos \angle ABC - \cos AC = 0;$$

$$\therefore \cos AC = \cos AB \cos BC + \sin AB \sin BC \cos ABC.$$

And, in the same manner, may the proposition be proved to be true of the cosines of the other sides of the triangle.

If, therefore, A, A', A'', be put for the three angles of a spherical triangle, and S, S', S'', for the three sides respectively opposite to them,

(I.) (1.) $\cos S = \cos S' \cos S'' + \sin S' \sin S'' \cos A$*.

(2.) $\cos S' = \cos S'' \cos S + \sin S'' \sin S \cos A'$.

(3.) $\cos S'' = \cos S \cos S' + \sin S \sin S' \cos A''$.

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* According to the system of notation here adopted, any one of the variations of the same Form may be deduced from any other, if the Form itself be perfectly general, by placing an additional accent over every letter which has not two accents, and wherever there are two accents, by suppressing them.

(231.) Cor. 1. If a, a', a'' be put for the angles, and s, s', s'' for the opposite sides of the polar triangle of ABC, then

$$\cos s = \cos s' \cos s'' + \sin s' \sin s'' \cos a;$$

that is, (Art. 78. and Introd. 19.)

$$-\cos A = -\cos A' \times -\cos A'' - \sin A' \sin A'' \cos S.$$
$$= \cos A' \cos A'' - \sin A' \sin A'' \cos S;$$

$$\therefore \text{(II.) (1.)} \cos A = \sin A' \sin A'' \cos S - \cos A' \cos A''.$$
$$\text{(2.)} \cos A' = \sin A'' \sin A \cos S' - \cos A'' \cos A.$$
$$\text{(3.)} \cos A'' = \sin A \sin A' \cos S'' - \cos A \cos A'.$$

(232.) Cor. 2. Hence, in a right-angled spherical triangle, having the angle A for a right angle, since $\cos A = 0$,

$$\cos S = \cos S' \cos S'' \quad \text{(Art. 230. I. 1.)}$$

$$\therefore \text{(III.)} \cos S' = \frac{\cos S}{\cos S''}, \cos S'' = \frac{\cos S}{\cos S'}.^*$$

Again, the angle A being a right angle, and therefore, $\cos A = 0$;

$$\therefore \text{(Art. 231. II.)} \sin A' \sin A'' \cos S - \cos A' \cos A'' = 0;$$

$$\therefore \cos S = \frac{\cos A'}{\sin A'} \cdot \frac{\cos A''}{\sin A''} = \text{(Introd. 17.)} \cot A' \cot A'';$$

$$\therefore \text{(IV.)} \cos S = \cot A' \cot A'' = \frac{\cot A''}{\tan A'} = \frac{1}{\tan A' \tan A''}$$

(Introd. 17.)

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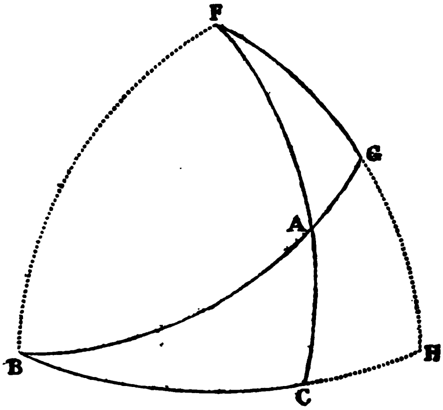
* In this, and the subsequent Forms, fractions are equated, rather than rectangles; because the quantities, under comparison, are thus more completely separated; and their mutual relation is the more readily and the more distinctly perceived. The implied theorems, also, are most easily read off, in words, as proportions, when the Forms, which represent them, are thus exhibited.

Next, if Form III. be applied to the right-angled tri-



angle AFG, which is the complemental triangle of ABC, right-angled at C, then (Art. 228. 229.)

$$\cos FG = \frac{\cos AF}{\cos AG}; \ \sin \angle ABC = \frac{\sin AC}{\sin AB}.$$

Hence, (V.) $\sin A' = \dfrac{\sin S'}{\sin S}$; and $\sin A'' = \dfrac{\sin S''}{\sin S}$,

the same application of Form III. being made to the other triangle, which is the complemental triangle of ABC.

And, if Form V. be applied to the triangle AFG,

$$\sin \angle FAG = \frac{\sin FG}{\sin AF} = \frac{\cos \angle ABC}{\cos AC}.$$

Hence, (VI.) $\sin A' = \dfrac{\cos A''}{\cos S''}$; $\sin A'' = \dfrac{\cos A'}{\cos S''}$.

Again, if Form IV. be applied to the complemental triangle AFG,

$$\cos FA = \frac{\cot \angle AFG}{\tan \angle FAG}; \text{ that is, } \sin AC = \frac{\tan BC}{\tan \angle BAC}.$$

Hence, (VII.) $\sin S' = \dfrac{\tan S''}{\tan A''}$; $\sin S'' = \dfrac{\tan S'}{\tan A'}$;

or, $\sin S' = \tan S'' \cot A''$; $\sin S'' = \tan S' \cot A'$.

And, if Form VII. be applied to the triangle AFG,

$$\sin FG = \frac{\tan AG}{\tan \angle AFG} = \frac{\cot \angle AFG}{\cot AG};$$

that is, $\cos ABC = \dfrac{\tan BC}{\tan AB}$.

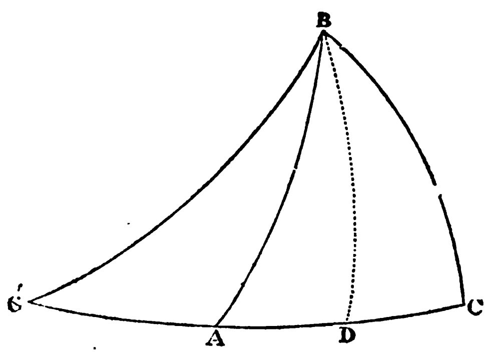
Hence, (VIII.) $\cos A' = \dfrac{\tan S''}{\tan S}$; $\cos A'' = \dfrac{\tan S'}{\tan S}$

or, $\cos A' = \tan S'' \cot S$; $\cos A'' = \tan S' \cot S$.

Prop. II.

(233.) *Theorem.* The sines of any two sides of a spherical triangle are to one another as the sines of the angles opposite to them.

Let ABC be a spherical triangle; and BC and BA any two of its sides: the sine of BC is to the sine of BA



as the sine of the angle A is to the sine of the angle C.

For draw (Art. 70.) from B the arch of a great circle BD at right angles to AC, and meeting AC, or AC produced, in D.

Then (Art. 232. V.) since BDA and BDC are right-angled spherical triangles, if unity be put for the radius,

$$\sin BD = \sin \angle C \sin BC = \sin \angle A \sin BA;$$

$$\therefore \sin BC : \sin BA :: \sin \angle A : \sin \angle C.$$

(234.) Cor. 1. Hence, if A, A', A'' be put for the angles of the triangle, and S, S', S'' for the sides respectively opposite to A, A', A'', then, (IX.)

$$(1.)\ \frac{\sin S}{\sin S'} = \frac{\sin A}{\sin A'}.\ (2.)\ \frac{\sin S'}{\sin S''} = \frac{\sin A'}{\sin A''}.\ (3.)\ \frac{\sin S''}{\sin S} = \frac{\sin A''}{\sin A}.$$

(235.) Cor. 2. If two spherical triangles have two angles of the one equal to two angles of the other, each to each, or an angle of the one being equal to an angle of the other, if two other angles, one in each triangle, be together equal to two right angles, in either case, the sines of the sides, about the third angles in each, shall be proportionals.

(236.) Cor. 3. If the two values of $\cos S'$, deducible from Art. 230. (I. 2. and 3.), be equated,

$$\cos S' = \cos S'' \cos S + \sin S'' \sin S \cos A'$$

$$= \frac{\cos S'' - \sin S \sin S' \cos A''}{\cos S};$$

$$\therefore \sin S'' \sin S \cos A'$$

$$=\frac{\cos S''-\sin S\sin S'\cos A''}{\cos S}-\cos S''\cos S$$

$$=\frac{\cos S''-\sin S\sin S'\cos A''-\cos S''\cos^2 S}{\cos S}$$

$$=\frac{\cos S''.(1-\cos^2 S)-\sin S\sin S'\cos A''}{\cos S};$$

$\therefore \cos S\sin S''\sin S\cos A'=\cos S''\sin^2 S-\sin S\sin S'\cos A''$ (Introd. 15.)

Dividing, then, both sides of the equation by $\sin S''\sin S$,

$$\cos S\cos A'=\frac{\cos S''}{\sin S''}.\sin S-\frac{\sin S'}{\sin S''}.\cos A''.$$

But (Introd. 17.)

$$\frac{\cos S''}{\sin S''}=\cot S''; \text{ and (Art. 223.) } \frac{\sin S'}{\sin S''}=\frac{\sin A'}{\sin A''}.$$

Wherefore, (X.)

(1.) $\cos S\cos A'=\cot S''\sin S-\sin A'\cot A''$.

(2.) $\cos S'\cos A''=\cot S\sin S'-\sin A''\cot A$.

(3.) $\cos S''\cos A=\cot S'\sin S''-\sin A\cot A'$.

(237.) SCHOLIUM. The Forms which are marked I. II. IX. X. are of themselves sufficient for all the common purposes of Spherical Trigonometry: and the intermediate Forms, are nothing more than those four general expressions, adapted to the particular case of right-angled spherical triangles. It is evident, also, that all the subsequent Forms may be considered as corollaries of the first proposition; they have here indeed been deduced from it, by means of the well-known properties of the

polar, or supplemental, and the complemental triangles: and this method of investigation has been adopted, because it appears the easiest for the learner to apprehend, and to retain. The same deductions may, however, be made, without any reference to the principles of Spherical Geometry, but in a different order.

Thus, first, Let a value of $\sin A = \sqrt{(1-\cos^2 A)}$ be obtained from Form I; so that $\sin A = F.\sin S$; $\sin A' = F.\sin S'$; $\sin A'' = F.\sin S''$; where F is the same function of S, S' and S'', in all the three cases; wherefore,

$$F = \frac{\sin A}{\sin S} = \frac{\sin A'}{\sin S'} = \frac{\sin A''}{\sin S''};$$

$$\text{or } \frac{\sin S}{\sin S'} = \frac{\sin A}{\sin A'}; \quad \frac{\sin S'}{\sin S''} = \frac{\sin A'}{\sin A''};$$

$$\frac{\sin S''}{\sin S} = \frac{\sin A''}{\sin A};$$

which is Form IX: and Form X. may be derived, as before, from Form I, by the help of Form IX, there being no reference, in that process, to any theorem of Spherical Geometry.

The second Form may next be deduced from Forms X. IX. and I. For from Forms X. 1. and IX. 2.

$$\cot A'' = \frac{\cot S'' \sin S - \cos S \cos A'}{\sin A'},$$

and, $$\sin A'' = \frac{\sin A' \sin S''}{\sin S'};$$

wherefore, if (Introd. 17.) $\cos. A''$ be put for $\sin A'' \cot A''$, the product of the two equations is,

$$\cos A'' = \frac{\cos S'' \sin S}{\sin S'} - \frac{\cos A' \sin S''}{\sin S'} . \cos S$$

$$= \frac{\cos S'' \sin A}{\sin A'} - \frac{\cos A' \sin S''}{\sin S'} . \cos S.$$

And, if two substitutions be further made, in this last value of $\cos A''$, so as to eliminate, first, $\cos S$, by putting for it its value from Form I. 1, and then $\cot S'$, introduced through that process, by putting, for it, its value from Form X. 3. there results the second Form,

$$\cos A'' = \sin A \sin A' \cos S'' - \cos A \cos A'$$
$$\cos A = \sin A' \sin A'' \cos S - \cos A' \cos A''$$
$$\cos A' = \sin A'' \sin A \cos S' - \cos A'' \cos A.$$

The four equations marked I. II. IX. X. having been thus obtained, the intermediate Forms may be deduced immediately from them, without the help of the complemental triangle. For, if $A = 90°$,

$$\cos S = \cos S' \cos S'' \quad \text{(I. 1.)}$$
$$\cos S = \cot A' \cot A'' \quad \text{(II. 1.)}$$
$$\sin A'' = \frac{\sin S''}{\sin S} \quad \text{(IX. 3.)}$$
$$\cos A'' = \sin A' \cos S'' \quad \text{(II. 3.)}$$
$$\cot A' = \cot S' \sin S'' \quad \text{(X. 3.)}$$
$$\text{or, } \tan S' = \tan A' \sin S''$$
$$\cos A'' = \cot S . \frac{\sin S'}{\cos S'} \quad \text{(X. 2.)}$$

that is, $\cos A'' = \cot S \tan S'$ (Introd. 17.)

The substance, therefore, of Spherical Trigonometry may be seen, at one view, in the narrow compass of this

Scholium. It is proper, however, to continue the developement of the more general theorems, in order to arrive at certain subsidiary propositions, which it is sometimes convenient to employ, in the actual solution of spherical triangles.

Prop. III.

(238.) *Theorem.* If an arch of a great circle drawn from the vertex of a spherical triangle cut the base at right angles,

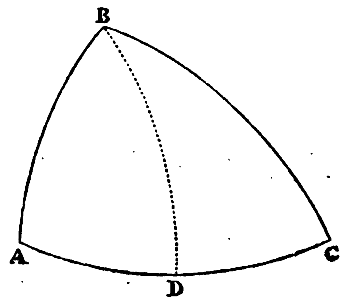
1. The sines of the segments of the base are to one another inversely as the tangents of the angles adjacent to them.

2. The cosines of the segments of the vertical angle are to one another inversely as the tangents of the sides to which they are adjacent.

3. The cosines of the segments of the base are to one another as the cosines of the sides adjacent to them.

4. The sines of the segments of the vertical angle are to one another as the cosines of the adjacent angles of the triangle.

Let the arch of a great circle *BD*, drawn from the



vertex B of the spherical triangle ABC cut the base AC at right angles: then,

1. Since, if unity be put for the radius, (Art. 233. VII.)

$$\tan BD = \tan \angle A \sin AD = \tan \angle C \sin CD;$$

$$\therefore \sin AD : \sin CD :: \tan \angle C : \tan \angle A.$$

2. Since, (Art. 232. VIII.)

$$\tan BD = \tan AB \cos \angle ABD = \tan BC \cos CBD;$$

$$\therefore \cos \angle ABD : \cos \angle CBD :: \tan BC : \tan BA.$$

3. Since (Art. 233. III.)

$$\cos BD = \frac{\cos AB}{\cos AD} = \frac{\cos BC}{\cos CD};$$

$$\therefore \cos AD : \cos CD :: \cos AB : \cos BC.$$

4. Lastly, since (Art. 233. VI.)

$$\cos BD = \frac{\cos \angle A}{\sin \angle ABD} = \frac{\cos \angle C}{\sin \angle CBD};$$

$$\therefore \sin \angle ABD : \sin \angle CBD :: \cos \angle A : \cos \angle C.$$

(239.) Cor. Hence, the same notation being used as in Art. 230, and V, V' being put for the segments of the vertical angle, and D, D' for the segments of the base respectively opposite to them, D being adjacent to the angle A, and D' to the angle A'

$$(\mathfrak{E})\ \frac{\sin D}{\sin D'} = \frac{\tan A'}{\tan A};\quad (\mathfrak{F})\ \frac{\cos V}{\cos V'} = \frac{\tan S}{\tan S'}.$$

$$(\mathfrak{G})\ \frac{\cos D}{\cos D'} = \frac{\cos S'}{\cos S};\quad (\mathfrak{H})\ \frac{\sin V}{\sin V'} = \frac{\cos A}{\cos A'}.$$

Prop. IV.

(240.) *Theorem.* If an arch of a great circle, drawn from the vertex of a spherical triangle, cut the base at right angles,

The tangent of the semi-base
Is to the tangent of the semi-sum of the sides
As the tangent of the semi-difference of the sides
Is to the tangent of the semi-difference of the segments of the base.

The same notation being used as in Art. 239. since,

$$\cos D : \cos D' :: \cos S' : \cos S;$$

$$\therefore \cos D + \cos D' : \cos D - \cos D' :: \cos S' + \cos S : \cos S' - \cos S;$$

$$\therefore \frac{\cos D - \cos D'}{\cos D + \cos D'} = \frac{\cos S' - \cos S}{\cos S' + \cos S}:$$

that is, (Introd. Art. 31.)

$$\frac{\tan \frac{1}{2}(D - D')}{\cot \frac{1}{2}(D + D')} = \frac{\tan \frac{1}{2}(S' - S)}{\cot \frac{1}{2}(S' + S)},$$

$$\text{or, } \frac{\tan \frac{1}{2}(D - D')}{\cot \frac{1}{2} S''} = \frac{\tan \frac{1}{2}(S' - S)}{\cot \frac{1}{2}(S' + S)};$$

$$\therefore (\mathfrak{I}) \ \frac{\tan \frac{1}{2} S''}{\tan \frac{1}{2}(S' + S)} = \frac{\tan \frac{1}{2}(S' - S)}{\tan \frac{1}{2}(D - D')}.$$

(241.) Cor. 1. By a similar process founded on Art. 238. 4. or else by means of the latter part of Art. 78, it may be shewn, that,

$$(\mathfrak{K})\ \frac{\tan\frac{1}{2}A''}{\cot\frac{1}{2}(A'+A)} = \frac{\tan\frac{1}{2}(V\sim V')}{\tan\frac{1}{2}(A'\sim A)},$$

$$\text{or, } \frac{\tan\frac{1}{2}(A'+A)}{\cot\frac{1}{2}A''} = \frac{\tan\frac{1}{2}(V\sim V')}{\tan\frac{1}{2}(A'\sim A)}.$$

(242.) Cor. 2. Similarly, since (Art. 238. 2),

$$\cos V : \cos V' :: \tan S : \tan S';$$

$$\therefore \frac{\tan S+\tan S'}{\tan S-\tan S'} = \frac{\cos V+\cos V'}{\cos V-\cos V'};$$

that is, (Introd. Art. 31.),

$$(\mathfrak{L})\ \frac{\sin(S+S')}{\sin(S-S')} = \frac{\cot\frac{1}{2}A''}{\tan\frac{1}{2}(V-V')}.$$

Prop. V.

(243.) *Theorem.* In any spherical triangle,

The tangent of the semi-sum of the two sides
Is to the tangent of their semi-difference
As the tangent of the semi-sum of the angles at the base
Is to the tangent of their semi-difference.

Let S and S' be the two sides, A and A' the two angles at the base.

Then (Art. 233.) $\sin S : \sin S' :: \sin A : \sin A'$;

$\therefore \sin S+\sin S' : \sin S-\sin S' :: \sin A+\sin A' : \sin A-\sin A'$;

$\therefore$ (Introd. Art. 31.)

$$(\mathfrak{M})\ \tan\tfrac{1}{2}(S+S') : \tan\tfrac{1}{2}(S-S') :: \tan\tfrac{1}{2}(A+A') : \tan\tfrac{1}{2}(A-A').$$

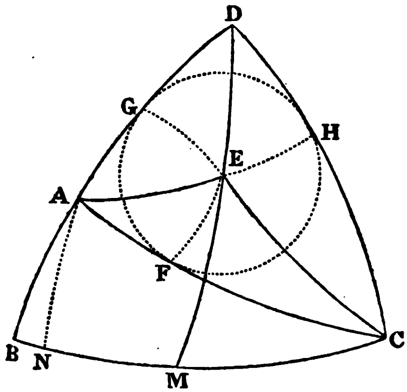
Prop. VI.

(244.) *Theorem.* In a scalene spherical triangle,

1. The sine of the semi-sum of any two sides
   Is to the sine of their semi-difference
   As the co-tangent of half of the angle included by them
   Is to the tangent of the semi-difference of the two other angles.

2. And, the cosine of the semi-sum of any two sides
   Is to the cosine of their semi-difference
   As the co-tangent of half of the angle included by them
   Is to the tangent of the semi-sum of the two other angles.

Let ABC be a scalene spherical triangle, of which AC and AB are any two sides; let the side AC be



greater than AB, and consequently, (Art. 129.) the angle ABC greater than ACB: then,

$\sin\frac{1}{2}(AC+AB) : \sin\frac{1}{2}(AC-AB) :: \cot\frac{1}{2}\angle BAC : \tan\frac{1}{2}(B-C)$,

nd $\cos\frac{1}{2}(AC+AB) : \cos\frac{1}{2}(AC-AB) :: \cot\frac{1}{2}\angle BAC : \tan\frac{1}{2}(B+C)$.

For, at the point C, in BC, make the angle BCD (Art. 96.) equal to ABC, and let CD meet BA produced, in D; wherefore, (Art. 105.) DB is equal to DC: bisect (Art. 101.) the angle BDC by the arch DM, which (Art. 109.) will also bisect BC at right angles in M: bisect, also, the angle DAC by the arch AE, which meets DM, in E; and join (Art. 66.) E, C: then (Art. 191.) EC bisects the angle ACD.

From E, draw (Art. 70.) the arches EF, EG, EH perpendicular to AC, AD and DC, respectively; the which perpendiculars (Art. 191.) are equal to one another.

If, therefore, A, B, C be the three angles of the triangle, it is evident, from the construction, that,

(1.) $\angle ACD = B - C$; and $FCE = \frac{1}{2}(B - C)$;

(2.) $\angle MCE = C + \frac{1}{2}(B - C) = \frac{1}{2}(B + C)$;

(3.) $\angle CAE = \frac{1}{2}(180° - A) = 90° - \frac{1}{2}A$;

also, $\angle HED =$ (Art. 116.) GED, and $\angle CEH = CEF$;

whence,

$\angle MEC + CEF = \angle MEF + 2.AEF = 2.\angle MEC + MEF$;

and $\therefore$ (4.) $\angle MEC = AEF$.

Again, (Art. 116.) $AG = AF$, and $GD = HD$; also, by the construction, $BD = CD$;

$$\therefore BA + AG = CH = CF = CA - AF;$$

$$\therefore BA = CA - AF - AG = CA - 2AF;$$

$$\therefore (5.)\ AF = \tfrac{1}{2}(AC - AB);$$

$$\therefore CF = AC - AF = AC - \tfrac{1}{2}(AC - AB);$$

$$\therefore (6.)\ CF = \tfrac{1}{2}(AC + AB).$$

But, (Art. 238. 1.) $\sin CF : \sin AF :: \tan \angle EAF : \tan \angle ECF$, that is,

$$\sin \tfrac{1}{2}(AC+AB) : \sin \tfrac{1}{2}(AC-AB) :: \cot \tfrac{1}{2} A : \tan \tfrac{1}{2}(B-C),$$

or (𝔑)

$$\sin \tfrac{1}{2}(S+S') : \sin \tfrac{1}{2}(S \sim S') :: \cot \tfrac{1}{2} A'' : \tan \tfrac{1}{2}(A \sim A').$$

Also (Art. 232. IV.) $\cos CE = \dfrac{\cot \angle CEM}{\tan \angle ECM} = (4.) \dfrac{\cot \angle AEF}{\tan \angle ECM}$,

$$\text{and } \cos AE = \frac{\cot \angle AEF}{\tan \angle FAE};$$

$$\therefore \frac{\cos CE}{\cos AE} = \frac{\tan \angle FAE}{\tan \angle ECM}; \text{ that is, (Art. 238. 3.)}$$

$$\frac{\cos CF}{\cos AF} = \frac{\tan \angle FAE}{\tan \angle ECM};$$

$$\therefore \cos CF : \cos AF :: \tan \angle FAE : \tan \angle ECM,$$

that is,

$$\cos \tfrac{1}{2}(AC+AB) : \cos \tfrac{1}{2}(AC-AB) :: \cot \tfrac{1}{2} A : \tan \tfrac{1}{2}(B+C),$$

or (𝔓) $\cos \tfrac{1}{2}(S+S') : \cos \tfrac{1}{2}(S \sim S') :: \cot \tfrac{1}{2} A'' : \tan \tfrac{1}{2}(A+A')$.

The proposition may also be deduced, from the preceding articles, without the aid of any geometrical construction.

For (Art. 243. 𝔐) $\dfrac{\tan \frac{1}{2}(S+S')}{\tan \frac{1}{2}(S \sim S')} = \dfrac{\tan \frac{1}{2}(A+A')}{\tan \frac{1}{2}(A \sim A')}$,

that is, (Art. 241. 𝔎)

$$\frac{\tan \frac{1}{2}(S+S')}{\tan \frac{1}{2}(S \sim S')} = \frac{\tan \frac{1}{2}(V \sim V')}{\tan^2 \frac{1}{2}(A' \sim A)} \cdot \cot \tfrac{1}{2} A''.$$

$$\text{Also, } \frac{\sin (S+S')}{\sin (S \sim S')} = \frac{\cot \frac{1}{2} A''}{\tan \frac{1}{2}(V \sim V')}. \text{ (Art. 242. 𝔏)}$$

And, if the two last equations be multiplied together, it follows, from Introd. 30, that,

$$\frac{\sin^2 \frac{1}{2}(S+S')}{\sin^2 \frac{1}{2}(S \sim S')} = \frac{\cot^2 \frac{1}{2} A''}{\tan^2 \frac{1}{2}(A \sim A')};$$

$$\therefore (\mathfrak{N}) \ \frac{\sin \frac{1}{2}(S+S')}{\sin \frac{1}{2}(S \sim S')} = \frac{\cot \frac{1}{2} A''}{\tan \frac{1}{2}(A \sim A')}.$$

Again, if the equation ($\mathfrak{N}$) be divided by the equation ($\mathfrak{M}$), it follows from Introd. 17, that

$$(\mathfrak{P}) \ \frac{\cos \frac{1}{2}(S+S')}{\cos \frac{1}{2}(S \sim S')} = \frac{\cot \frac{1}{2} A''}{\tan \frac{1}{2}(A+A')}.$$

(245.) Cor. If Forms ($\mathfrak{N}$) and ($\mathfrak{P}$) be applied to the polar triangle, there result the two following proportions,

$(\mathfrak{Q}) \sin \frac{1}{2}(A+A') : \sin \frac{1}{2}(A \sim A') :: \tan \frac{1}{2} S'' : \tan \frac{1}{2}(S \sim S'),$

$(\mathfrak{R}) \cos \frac{1}{2}(A+A') : \cos \frac{1}{2}(A \sim A') :: \tan \frac{1}{2} S'' : \tan \frac{1}{2}(S+S')$*.

Prop. VII.

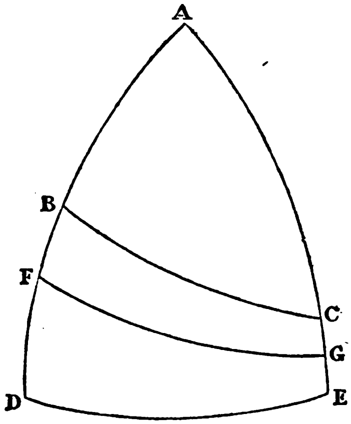
(246.) ***Problem.*** The three sides of a spherical triangle being given, to find the value of a function of the spherical distance, between a given point in one of their sides, and a given point in either of the other two.

Let AFG be the given triangle; also let B be a given point, in any of its sides, as AF; let C be a given point, in either of the other sides, as AG: and let B, C be joined. It is required to express a function of BC, in terms of the functions of the several given arches.

If Form I. (Art. 230.) be applied, severally, to the two spherical triangles ABC, AFG,

* The proportions marked ($\mathfrak{N}$), ($\mathfrak{P}$), ($\mathfrak{Q}$), ($\mathfrak{R}$), are called, from their inventor, *Naper's Analogies.*

$$\cos \angle A = \frac{\cos FG - \cos AF \cos AG}{\sin AF \sin AG} = \frac{\cos BC - \cos AB \cos AC}{\sin AB \sin AC};$$



$$\therefore \cos BC = \frac{(\cos FG - \cos AF \cos AG) \cdot \sin AB \sin AC}{\sin AF \sin AG} +$$
$\cos AB \cos AC.$

(247.) Cor. 1. In the same manner, an expression may be found, for computing the spherical distance DE, between two given points D and E, in the two sides AF and AG produced.

(248.) Cor. 2. If the point C coincide with the point G,

$$\cos GB = (\cos FG - \cos AF \cos AG) \cdot \frac{\sin AB}{\sin AF} + \cos AB$$
$\cos AG.$

Prop. VIII.

(249.) *Problem.* If three spherical triangles stand upon the same base, and upon the same side of it, and

have the three sides terminated in the one extremity equal; and if there be given the three sides terminated in the other extremity, and the differences of the three angles which they severally make with the base; to find the values of functions of the common base, of the equal sides, and of the angles, of which the differences are given.

Let S' be put for the common base, S for each of the three equal sides of the proposed triangles S'', S''', S'''' for the three given sides; A for the least of the angles opposite to the side S; and $(A+a)$, $(A+b)$, for the two other angles at the same extremity of the base, a and b being the given differences.

Then (Art. 230. I.)

$$\cos S = \cos S' \cos S'' + \sin S' \sin S'' \cos A,$$
$$\cos S = \cos S' \cos S''' + \sin S' \sin S''' \cos (A+a),$$
$$\cos S = \cos S' \cos S'''' + \sin S' \sin S'''' \cos (A+b);$$

$$\therefore \frac{\cos S}{\cos S'} = \cos S'' + \sin S'' \ (\cos A . \tan S')$$

$$= \cos S''' + \sin S''' \cos a (\cos A \tan S') - \sin S''' \sin a (\sin A \tan S')$$

(Introd. 28.)

$$= \cos S'''' + \sin S'''' \cos b (\cos A \tan S') - \sin S'''' \sin b (\sin A \tan S').$$

And, since the above three equations involve no more unknown quantities than the functions of the three quantities which are required to be found, it is evident, that the three unknown quantities may be, each of them, eliminated, by the usual processes of algebraical calculation, and by substituting (Introd. 17.) $\frac{1}{\sqrt{(1+\tan^2 S')}}$ for $\cos S'$: and the equations will be the more easily solved if $\frac{\cos S}{\cos S'}$,

$\cos A \tan S'$, and $\sin A \tan S'$, be first considered as the unknown quantities.

(250.) Cor. If S'', S''', and the angles A and $(A+a)$ be given, the tangent of S' may be found in terms of the functions of the given quantities by subtracting the second of the first three equations, in Art. 249, from the first: for then,

$$S'(\cos S'' - \cos S''') + \sin S'(\sin S'' \cos A - \sin S''' \cos (A+a)) = 0;$$

$$\therefore \text{(Introd. 17.)} \tan S' = \frac{\cos S''' - \cos S''}{\sin S'' \cos A - \sin S''' \cos (A+a)}.$$

(251.) Scholium. The two preceding articles contain so much, as properly belongs to Spherics, of the solutions of two astronomical problems, which may be thus enunciated.

1. Three known Stars having been observed in the same almacantar, to find the latitude of the place of observation, the hour, and the common zenith-distance of the three Stars.

2. Two known Stars having been observed, at a given hour, in the same almacantar, to find the latitude of the place of observation.

The process, by which the expressions, investigated in Art. 249. 250, are adapted to logarithmical calculation, may be seen in the first Volume of Delambre's Astronomy.

PART II.

THE ELEMENTS OF

Spherical Trigonometry.

SECTION II.

ON THE SOLUTION OF RIGHT-ANGLED AND QUADRANTAL SPHERICAL TRIANGLES.

DEFINITION.

(252.) THE three sides, and the three angles, are called the *Parts* of a Spherical Triangle.

(253.) DEF. Any three of the six parts of a proposed spherical triangle being given, the finding of the remaining three, or of their trigonometrical functions, is called the *Solution* of that triangle: and when, by the help of Spherical Trigonometry, the three parts, or their functions, which were, at the first, unknown, have been found, the triangle is said to be *solved*.

(254.) COR. 1. A spherical triangle, having all its angles right angles, and consequently, (Art. 122.) all its

sides quadrants, is solved without the help of trigonometry.

(255.) Cor. 2. If two of the angles of a spherical triangle be right angles, or two of its sides quadrants, then, of the remaining side and angle, if the one be given, the other (Art. 121. and 54.) is known; and the triangle is, in that case, also, solved, without the help of trigonometry.

Prop. I.

(256.) *Problem.* Two parts, besides the right angle, being given, of a spherical triangle, which has only one of its angles a right angle, to solve the triangle.

Let A denote the right angle, and S the hypotenuse; A', A'' the two oblique angles; and S', S'' the two sides respectively opposite to them.

Case 1.

Let the hypotenuse, S, and either of the other sides, S' or S'', be given.

(1.)

$$\sin A' = \frac{\sin S'}{\sin S} \quad \text{(Art. 232. V.)}$$

$$\cos A'' = \tan S' \cot S \quad \text{(Art. 232. VIII.)}$$

$$\cos S'' = \frac{\cos S}{\cos S'}, \quad \text{(Art. 232. III.)}$$

(2.)

$$\sin A'' = \frac{\sin S''}{\sin S}$$

$$\cos A' = \tan S'' \cot S$$

$$\cos S' = \frac{\cos S}{\cos S''}.$$

Case 2.

Let the hypotenuse S, and either of the two oblique angles A' or A'', be given.

(1.)

$$\sin S' = \sin S \sin A'. \quad (232.\ \text{V.})$$

$$\tan S'' = \tan S \cos A' \quad (\text{VIII.})$$

$$\cot A'' = \cos S \tan A' \quad (\text{IV.})$$

(2.)

$$\sin S'' = \sin S \sin A''.$$

$$\tan S' = \tan S \cos A''.$$

$$\cot A' = \cos S \tan A''.$$

Case 3.

Let the two sides, S' and S'', be given.

$$\tan A' = \frac{\tan S'}{\sin S''} \quad (\text{Art. } 232.\ \text{VII.})$$

$$\tan A'' = \frac{\tan S''}{\sin S'} \quad (\text{VII.})$$

$$\cos S = \cos S' \cos S'' \quad (\text{III.})$$

Case 4.

Let S' and A'', or S'' and A', be given.

(1.)

$$\tan S = \frac{\tan S'}{\cos A''} \quad (232.\ \text{VIII.})$$

$$\tan S'' = \sin S' \tan A'' \quad (232.\ \text{VII.})$$

$$\cos A' = \sin A'' \cos S'.$$

(2.)

$$\tan S = \frac{\tan S''}{\cos A'}$$

$$\tan S' = \sin S'' \tan A'$$

$$\cos A'' = \sin A' \cos S''.$$

Case 5.

Let S' and A', or S'' and A'', be given.

(1.

$$\sin S = \frac{\sin S'}{\sin A'} \quad (232.\ \text{V.})$$

$$\sin S'' = \tan S' \cot A' \quad (232.\ \text{VII.})$$

$$\sin A'' = \frac{\cos A'}{\cos S'}, \quad (332.\ \text{VI.})$$

(2.)

$$\sin S = \frac{\sin S''}{\sin A''}$$

$$\sin S' = \tan S'' \cos A''$$

$$\sin A' = \frac{\cos A''}{\cos S''}.$$

CASE 6.

Let A' and A'' be given.

$$\cos S = \cot A' \cot A'' \quad (232.\ \text{IV.})$$

$$\cos S' = \frac{\cos A'}{\sin A''} \quad (232.\ \text{VI.})$$

$$\cos S'' = \frac{\cos A''}{\sin A'} \quad (232.\ \text{VI.})$$

(257.) COR. 1. Hence, any proposed isosceles spherical triangle may be solved, if any three of its six parts be given.

For, its solution is reduced to that of a right-angled triangle, by drawing, from the vertex, an arch of a great circle (Art. 70.) perpendicular to the base: because (Art. 107.) the angles at the base are equal, and (Art. 109.) the arch, which is perpendicular to the base, bisects the base and the vertical angle.

(258.) COR. 2. If (S) be the given side of an equilateral spherical triangle, and (A) any one of its equal angles (Art. 108.) the angle A may be found by means of Art. 232. VIII. a perpendicular arch having been first let fall, from either of the other angles, upon the opposite side. For, then, since that side is bisected (Art. 109.) by the perpendicular arch, $\cos A = \tan \frac{1}{2} S \cot S$.

(259.) COR. 3. If two of the sides, or two of the angles, of a spherical triangle, be the supplements, each of the other, it is evident, from Art. 210, that the solution of such a triangle may be reduced to that of an isosceles

spherical triangle; which latter solution, as hath been shewn, (Art. 257.) may be made to depend on the solution of a right-angled spherical triangle.

(260.) SCHOLIUM. The five parts exclusive of the right angle, of a right-angled spherical triangle, admit of $\left(\frac{5.4.3}{1.2.3}\right)$ ten combinations, when they are taken three and three together: but, of these, only six are essentially different from each other. Now, every one of these six different combinations comprehends three problems; for any two of the quantities, out of each set of three, may be the quantities that are given. There are, therefore, in the whole, eighteen such problems. They cannot, all of them, however, be considered as really different problems; because, in some of the combinations, two of the three problems, when they are stated in general terms, become, in effect, one and the same problem.

Again, the parts, which are sought, are found in terms of their trigonometrical functions. Whenever, therefore, any required part is expressed in terms of its sine, the part itself (Introd. 14. 19.) is only so far determined, as to be known to be one of two different arches, or angles, which are supplements of each other: so that, in reality, there are then two different triangles which satisfy the general conditions of the problem. This is the case, when either the base, or the perpendicular, and the opposite angle, are given, to find (V. VI. VII.) the other parts. There remains, therefore, in this case, an am-

biguity; unless it can be removed, by the application of Art. 127. 129.

But, in all the other cases, in which the unknown part is expressed in terms of its cosine, of its tangent, or of its co-tangent, the *sign* of the value of that function will shew whether the part sought be greater, or less, than a quadrant, or than a right angle. It will be less (Introd. 19.) than that quantity, if the value of its cosine, its tangent, or its co-tangent, be positive, and greater, when that value is negative. Here, therefore, the ambiguity, which (Introd. 14.) might otherwise obtain, is removed by the algebraic sign of the function : and it may, also, be removed, by the application of Art. 130.

The following Table may be used as a key to the solution of all the cases of right-angled spherical triangles: Any two quantities, in each of the six specified combinations, being given, the third may be found, by the Forms and articles to which a reference is made.

(1.) Hypotenuse, and the other two Sides
(III. and Art. 130.)

(2.) Hypotenuse and two oblique Angles
(IV. and Art. 130.)

(3.) Hypotenuse, one other Side and adjacent Angle
(VIII. and Art. 130.)

(4.) Hypotenuse, one other Side and opposite Angle
(V. and Art. 127. or else ambiguous.)

(5.) Base, Perpendicular and either oblique Angle
(VII. and Art. 127. or else ambiguous.)

(6.) Two oblique Angles, and Base, or Perpendicular (VI. and Art. 127. or else ambiguous.)

It may, further, be remarked, that, when a right-angled spherical triangle is proposed, for solution, the Forms which have been investigated, serve, also, to determine whether the problem be possible. The following conclusions may easily be derived from them.

In right-angled spherical triangles, having only one angle a right angle,

1. The complement of the hypotenuse cannot exceed the complement of either of the other sides (III.)
2. The hypotenuse may have any magnitude, less than 180°, relative to that of either of the oblique angles (IV.)
3. The complement of either of the two oblique angles cannot be less than the complement of the opposite side (V. or VI.)
4. Either of the two oblique angles may have any magnitude, less than 180°, with respect to the side adjacent to it (VIII.)
5. But the two oblique angles must be, together, greater than a right angle (VI. or Art. 82.)
6. The two sides, containing the right angle, may have any relative magnitude.

In consequence of the want of absolute exactness, in the logarithmic Tables, which are in use, the Forms that have been investigated for the purpose of solving right-

angled spherical triangles, will not yield results sufficiently correct, if the sines and the cosines of the required parts be very great; that is, if they approximate to the value of the radius. In these cases, therefore, it becomes necessary to substitute other equivalent expressions, which may exhibit the parts, that are sought, in terms of their tangents.

Prop. II.

(261.) *Problem.* The value of the sine, or the cosine, of any required part, of a right-angled spherical triangle, having been found in terms of the given parts, to find the value of the tangent, or of the co-tangent, of the half of that part.

Let A denote the right angle, and S the hypotenuse; A' one of the other angles, and S' the side opposite to it; A'' the third angle, and S'' the third side, of a right-angled spherical triangle; and let *unity* be put for the radius: then, (Art. 232. IV.)

$$\cos S = \cot A' \cot A'';$$

and (Introd. Art. 30.) $\tan^2 \frac{1}{2} S = \dfrac{1 - \cos S}{1 + \cos S}$;

$$\therefore \tan^2 \tfrac{1}{2} S = \frac{1 - \cot A' \cot A''}{1 + \cot A' \cot A''}$$

$$= \frac{\sin A' \sin A'' - \cos A' \cos A''}{\sin A' \sin A'' + \cos A' \cos A''} \text{ (Introd. 30.)}$$

$$= \frac{-\cos (A + A'')}{\cos (A' \sim A'')} \text{ (Introd. 28.)}$$

$$\therefore (\mathfrak{A}) \tan \tfrac{1}{2} S = \sqrt{\frac{-\cos(A'+A'')}{\cos(A' \sim A'')}}\ *.$$

Again, since (Art. 232. VI.) $\cos S'' = \dfrac{\cos A''}{\sin A'}$,

$$\therefore \tan^2 \tfrac{1}{2} S'' = \frac{1-\cos S''}{1+\cos S''} = \frac{1-\dfrac{\cos A''}{\sin A'}}{1+\dfrac{\cos A''}{\sin A'}} = \frac{\sin A' - \sin B}{\sin A' + \sin B},$$

(if $B = 90 - A''$)

$$= \frac{\tan \frac{1}{2}(A'-B)}{\tan \frac{1}{2}(A'+B)} \text{ (Introd. 31.)}$$

$$= \tan \tfrac{1}{2}(A'-B) \cot \tfrac{1}{2}(A'+B)$$

$$= \tan \frac{A'+A''-90}{2} \cot \frac{A'-A''+90}{2};$$

$$\therefore (\mathfrak{B}) \tan \tfrac{1}{2} S'' = \left(\tan. \frac{A'+A''-90}{2} . \cot \frac{A'-A''+90}{2}\right)^{\frac{1}{2}}.$$

Next, since (Art. 232. VIII.) $\cos. A'' = \dfrac{\tan S'}{\tan S}$;

$$\therefore \tan^2 \tfrac{1}{2} A'' = \frac{1-\cos A''}{1+\cos A''} = \frac{1-\dfrac{\tan S'}{\tan S}}{1+\dfrac{\tan S'}{\tan S}}$$

$$= \frac{\tan S - \tan S'}{\tan S + \tan S'}$$

$$= \frac{\sin(S-S')}{\sin(S+S')} \text{ (Introd. 31.)}$$

* It follows, from the equation marked ($\mathfrak{A}$), that the hypotenuse of a right-angled spherical triangle may be found, if the cosines of the sum, and of the difference, of the two oblique angles be known, even when those angles themselves are not known.

$$\therefore \ (\mathfrak{C}) \ \tan \tfrac{1}{2} A'' = \sqrt{\frac{\sin(S-S')}{\sin(S+S')}}.$$

Also, (Art. 232. III.) $\cos S'' = \dfrac{\cos S}{\cos S'}$;

$$\therefore \ \tan^2 \tfrac{1}{2} S'' = \frac{1-\cos S''}{1+\cos S''} = \frac{1-\dfrac{\cos S}{\cos S'}}{1+\dfrac{\cos S}{\cos S'}}$$

$$= \frac{\cos S' - \cos S}{\cos S' + \cos S}$$

$$= \tan \tfrac{1}{2}(S+S') \tan \tfrac{1}{2}(S-S') \ \text{(Introd. 31.)}$$

$$\therefore \ (\mathfrak{D}) \ \tan \tfrac{1}{2} S'' = \sqrt{[\tan \tfrac{1}{2}(S+S') . \tan \tfrac{1}{2}(S-S')]}.$$

Lastly, (Art. 232. V.) $\sin S' = \sin S \sin A'$.

Let, therefore, $\sin S' = \sin (90^\circ - 2z) = \cos 2z$

$$= \frac{1-\tan^2 z}{1+\tan^2 z} \ \text{(Introd. Art. 30.)}$$

whence, $\tan^2 z = \dfrac{1-\sin S'}{1+\sin S'}$, or $\dfrac{1-\sin S \sin A'}{1+\sin S \sin A'}$ (Art. 232. V.)

$= \dfrac{1-\tan x}{1+\tan x}$ (x being such that $\tan x = \sin S \sin A'$)

$$= \tan (45^\circ - x) . \ \text{(Introd. 29. 11.)}$$

If, therefore, x be first found, from the assumed equation, $\tan x = \sin S \sin A'$, and if z be next determined, from the equation, $\tan z = \sqrt{(45^\circ - x)}$, the side S', which is equal to $90^\circ - 2z$, will be known.

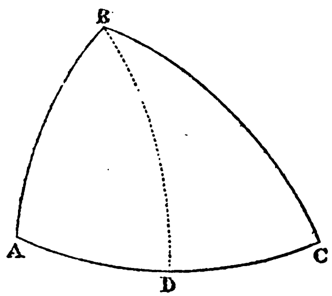
Prop. III.

(262.) *Problem.* Two parts, besides the quadrant, of a quadrantal triangle, which has only one of its sides a quadrant, being given, to solve the triangle.

It is evident, that the six theorems which have been applied (Art. 256.) to the solution of right-angled spherical triangles, may be readily translated, so as to express the relations existing between the parts of a quadrantal triangle, by means of Art. 78. For the polar triangle, in this case, is a quadrantal triangle. The four general Forms, also, which are marked (I.) (II.) (IX.) (X.) will give the same results, if a side in each of them be made equal to a quadrant. Thus,

If $S=90^\circ$; $\sin A' = \sin A \sin S'$ (IX.)
and $\sin A'' = \sin A \sin S''$.
$S''=90^\circ$: $\cos A'' = -\cot S \cot S'$ (I.)
$S=90^\circ$; $\cos S'' = \cos A'' \sin S'$ (I.)
$S''=90^\circ$; $\cos A'' = -\cos A \cos A'$ (II.)
$S=90^\circ$; $\cot S'' = \sin A' \cot A''$ (X.)
$S''=90^\circ$; $\cos S = -\tan A' \cot A''$ (X.)

Or, let CBD be a quadrantal triangle, having the side CB, and none other of its sides, a quadrant: from



C as a pole, at the distance CB, let there be described (Art. 59.) the great circle BA, meeting CD produced, in

A; wherefore, (Art. 50.) the angles *CBA*, *CAB*, are right angles, *BD* is common to the two triangles *CBD*, *BAD*; *AD* is the complement of *DC*; the angle *ABD* is the complement of *DBC*; and *BA* measures (Art. 54.) the angle *C*. It is manifest, then, that if two parts, besides the quadrant *BC*, of the triangle *CBD*, be given, the triangle *BAD* may be solved (Art. 256.): and if all the parts of the triangle *ABD* be known, all the parts of *CBD* are thereby known.

PART II.

THE ELEMENTS OF

Spherical Trigonometry.

SECTION III.

ON THE SOLUTION OF OBLIQUE-ANGLED SPHERICAL TRIANGLES.

(263.) *Problem.* THREE of the six parts of an oblique-angled spherical triangle being given, to solve the triangle.

CASE 1.

Let the three sides, S, S', S'', be given, to find the three angles, A, A', A'', of the triangle.

Then, (Art. 230. I.) $\cos A'' = \dfrac{\cos S''}{\sin S \sin S'} - \dfrac{\cos S \cos S'}{\sin S \sin S'}$

$$= \frac{\cos S''}{\sin S \sin S'} - \cot S . \cot S'$$

(Introd. 17.)

Find, therefore, two numbers m and n, such that

$$m = \frac{\cos S''}{\sin S \sin S'};$$

$$\text{and } n = \cot S \cot S':$$

Then, $\cos A'' = m - n.$

And, from this last equation, the angle A'' is determined.

Or, since (Art. 230. I.)

$$\cos A'' = \frac{\cos S'' - \cos S \cos S}{\sin S \sin S'}, \text{ therefore (Introd. 30.)}$$

$$2 \sin^2 \tfrac{1}{2} A'' = 1 - \cos A'' = \frac{\sin S \sin S' - \cos S'' + \cos S \cos S'}{\sin S \sin S'}$$

$$= \frac{\cos (S' - S) - \cos S''}{\sin S \sin S'} \text{ (Introd. 28.)}$$

$$= \frac{2 \sin \frac{1}{2} (S'' - S' + S) \sin \frac{1}{2} (S'' + S' - S)}{\sin S \,.\, \sin S'} \text{ (Introd. 31.)}$$

Hence, if P be put for the semi-sum of the three sides of the triangle,

$$\sin^2 \tfrac{1}{2} A'' = \frac{\sin (P - S') \sin (P - S)}{\sin S \sin S'}.$$

In the same manner, by *adding* the value of $\cos A''$ to *unity*, it may be shewn that,

$$\cos^2 \tfrac{1}{2} A'' = \frac{\sin (P - S'') \sin P}{\sin S \sin S'}.$$

And, if the former of the two equations, last deduced, be divided by the latter, then, (Introd. 17.)

$$\frac{\sin^2 \frac{1}{2} A''}{\cos^2 \frac{1}{2} A''} = \tan^2 \tfrac{1}{2} A'' = \frac{\sin (P - S') \,.\, \sin (P - S)}{\sin P \,.\, \sin (P - S'')}.$$

After having, by one of the above four methods, found the angle A'', the other angles may readily be found, by Art. 234. IX: and, since (Art. 129.) the greater side is opposite to the greater angle, the *order* of the angles, according to their magnitude, will be known; and it will appear, also, whether any one of them be obtuse.

The problem proposed, in this first case, is always *possible*, if any two of the three given sides be greater than the third, and if each of them be less than 180°: neither is there any ambiguity, in this case, in the answer obtained *.

Case 2.

Let the three angles, A, A', A'', be given, to find the three sides, S, S', S'', of the triangle.

Then (Art. 231. II.)

$$\cos S'' = \frac{\cos A'' + \cos A \cos A'}{\sin A \sin A'}$$

$$= \frac{\cot A'' + \dfrac{\cos A \cos A'}{\sin A''}}{\dfrac{\sin A \sin A'}{\sin A''}} \quad \text{(Introd. 17.)}$$

And, if $\dfrac{\cos A \cos A'}{\sin A''} = \tan y$, then

* It is manifest, that to this first Case, of the solution of oblique-angled spherical triangles, may be reduced the following problem: "The inclinations, to one another, of three straight lines, that are not in the same plane, being given, to find the angle made by the *projections*, of any two of the straight lines, on a plane that is perpendicular to the third."

$$\cos S'' = \frac{\tan(90^\circ - A'') + \tan y}{\dfrac{\sin A \sin A'}{\sin A''}}$$

$$= \frac{\sin(90^\circ - A'' + y)}{\sin A'' \cos y \dfrac{\sin A \sin A'}{\sin A''}} \text{ (Introd. 32.)}$$

$$= \frac{\cos(A'' - y)}{\sin A \sin A' \cos y}.$$

First, therefore, find y, from the equation, $\tan y = \dfrac{\cos A \cos A'}{\sin A''}$; and S'' will then be known from the final equation.

Or, by finding the sum, and the difference, of unity, and the value of $\cos S''$, as it is given in Art. 231. II. and then, by dividing the resulting equations, as in the preceding case, if p be put for the semi-sum of the three given angles,

$$\sin^2 \tfrac{1}{2} S'' = \frac{-\cos p \cos(p - A'')}{\sin A \sin A'}$$

$$\cos^2 \tfrac{1}{2} S'' = \frac{\cos.(p - A) \cos(p - A')}{\sin A \sin A'}$$

$$\tan^2 \tfrac{1}{2} S'' = \frac{-\cos p . \cos(p - A'')}{\cos(p - A) \cos(p - A')}.$$

Which expressions might also have been deduced from the solutions of the first case, by means of the polar triangle.

When S'' has been found, the other sides may be computed, by Art. 234. IX.

In this second case, the problem proposed is always *possible*, if the aggregate of the three given angles be less than six right angles, and greater than two right angles: and the problem admits of only one answer.

CASE 3.

Let any two sides as S and S', and the included angle A'', be given, to find the other parts of the triangle.

Then (Art. 230. I.)

$$\cos S'' = \cos S \cos S' + \sin S \sin S' \cos A''.$$

Find, therefore, two numbers m and n, such that

$$m = \cos S \cos S',$$

$$\text{and } n = \sin S \sin S' \cos A'';$$

$$\text{then, } \cos S'' = m + n.$$

Or, since

$$\cos S'' = \cos S \cos S' + \sin S \sin S' \cos A''$$
$$= \cos S (\cos S' + \cos A'' \tan S \sin S') \text{ (Introd. 17.)}$$

if, therefore,

$$\tan x = \cos A'' \tan S,$$

$$\cos S'' = \cos S (\cos S' + \tan x \sin S')$$

$$= \frac{\cos S}{\cos x}(\cos S' \cos x + \sin x \sin S') \text{ (Introd. 17.)}$$

$$= \frac{\cos S . \cos (S' - x)}{\cos x}.\text{* (Introd. 28.)}$$

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* It is evident, from Art. 232. VIII. that x is the base of a right-angled spherical triangle, of which S is the hypotenuse, and A'' an angle at the base. So that the problem is solved by drawing an arch, at right angles to one of the known sides, from the opposite angle.

And, when the third side has thus been computed, the angles, which are sought, may be found, by means of Art. 234. IX.

Or, the two unknown angles may, both of them, be first found, by combining their sum and difference, which are to be derived from Naper's Analogies, (Art. 244.); and the third side may then be found by means of Art. 234. IX.

Or, if the only part, that is sought, be A'', one of the unknown angles, then (Art. 236. X.)

$$\cos S \cos A' = \cot S'' \sin S - \sin A' \cot A'';$$

$$\therefore \cot A'' = \frac{\cot S'' \sin S - \cos S \cos A'}{\sin A'}$$

$$= \frac{\cot S'' \sin S}{\sin A'} - \cos S \cot A' \quad \text{(Introd. 17.)}$$

$$= \frac{\cot S'' \sin S}{\sin A'} - \cos S \cot A'.$$

Therefore, $\cot A''$ may be computed, either by finding two numbers, m and n, for the two separate parts of its value; or by drawing, from the extremity of one of the known sides, an arch of a great circle perpendicular to the other known side; by which the proposed triangle is divided into two right-angled spherical triangles. The third side, if it were required, might then be found, by Art. 238. 3; and the remaining angle, by Art. 234. IX.

In this third case, the problem is always *possible* (Art. 16.); the given sides, and the given angle, being, each of them, of any magnitude less than 180°.

CASE 4.

Let there be given any one of the sides, as S'', and the two angles adjacent to it, A, A', to find the other parts of the triangle.

Then (Art. 231. II.)

$$\cos A'' = \cos S'' \sin A \sin A' - \cos A \cos A'.$$

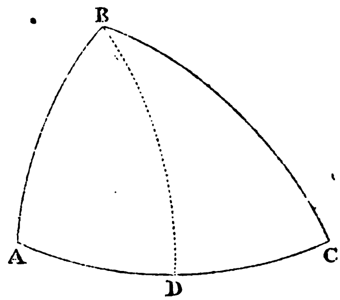
Let $\cos S'' \sin A \sin A' = \sin x$; and $\cos A \cos A' = \sin y$: so that x and y may be considered as known quantities;

$$\therefore \cos A'' = \sin x - \sin y = 2 \sin \frac{x-y}{2} \cos \frac{x+y}{2} \text{ (Introd. 31.)}$$

or, let $\cos S'' \sin A \sin A' = \tan z$; and $\cos A \cos A' = \tan u$.

$$\text{Then, } \cos A'' = \tan z - \tan u = \frac{\sin(z-u)}{\cos z \cos u} \text{ (Introd. 32.)}$$

Or, let ABC be the proposed spherical triangle, and let the arch BD be drawn from B, at right angles to the



side AC; let the angle $BAC = A$; the angle $ABC = A'$; the side $AB = S''$; and let A, A' and S'' be the given parts; let, also, the angle $ABD = x$:

Then, (Art. 232. IV.) $\cot x = \cos S'' \tan A$.

And (Art. 238. 3.) $\cos A'' = \dfrac{\sin (A' - x)}{\sin x} . \cos A$.

Whence, A'' may be said to be found.

Or, the two unknown sides may first be deduced, from the values of their sum and difference; which values may be found by means of Naper's Analogies [Art. 245. (*Q*) (*R*)].

When the third angle, or one of the unknown sides, has thus been found, the other unknown parts of the triangle may be found by means of Art. 234. IX.

This case is always *possible*, under the same restrictions as the preceding case.

Case 5.

Let any two sides, as S and S'', and the angle opposite to one of them, as A'', be given, to find the other parts of the triangle.

Then, (Art. 236. X.)

$$\cot S'' \sin S = \cot A'' \sin A' + \cos S \cos A';$$

$$\therefore \cot S'' . \frac{\sin S}{\cos S} = \frac{\cot A'' \sin A'}{\cos S} + \cos A'.$$

$$\text{Let } \frac{\cot A''}{\cos S} = \tan x;$$

$$\therefore \cot S'' \frac{\sin S}{\cos S} = \cot S'' \tan S = \tan x \sin A' + \cos A'$$

$$= \frac{\sin x \sin A' + \cos x \cos A'}{\cos x}$$

(Introd. 17.)

$$= \frac{\cos (A' - x)}{\cos x} \text{ (Introd. 28.)}$$

$$\therefore \cos(A' - x) = \cos x \tan S \cot S''.$$

Whence $A' - x$ is known; and $A' = (A' - x) + x$.

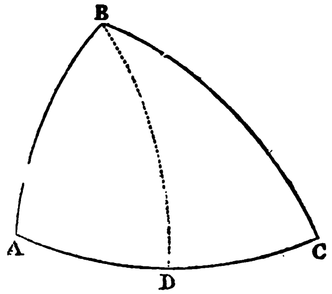
But, if for $\frac{\cot A''}{\cos S} = \tan x$, there be substituted the equivalent equation, $\frac{1}{\tan A'' \cos S} = \frac{1}{\cot x}$, or $\cot x = \cos S \tan A''$, then,

$$\cos(x - A') = \cos x \tan S \cot S'';$$

$$\text{and } A' = x - (x - A').$$

If both these expressions give a positive value for A', the general problem, proposed in this case, admits of two solutions; but, if either of the expressions give a negative value, that value of A' must be rejected.

Or, let ABC be the proposed triangle; and, in this case, let the perpendicular arch BD fall on AC (S') the



unknown side; let $AB = S''$; $CD = x$; and let S' be first sought.

Then, (Art. 234. VIII.) $\tan x = \cos A'' \tan S$;

and, (Art. 238. 4.) $\cos(S' - x) = \frac{\cos S''}{\cos S} \cdot \cos x$.

Whence, $S'-x$ may be said to be known; and, $S' = (S' - x) + x$.

It is still, however, uncertain, whether the value of $\frac{\cos S'' . \cos x}{\cos S}$ belong to $S'-x$, or to $x-S'$.

The third angle, or the third side may next be found, by means of Art. 234. IX.

Or, (Art. 234. IX.) $\sin A = \frac{\sin S}{\sin S''} . \sin A''$; whence A is known.

Then, $\tan \frac{1}{2} S' = \frac{\cos \frac{1}{2} (A + A'')}{\cos \frac{1}{2} (A - A'')} \tan \frac{1}{2} (S + '')S$ (Art. 245.)

and, $\tan \frac{1}{2} A' = \frac{\cos \frac{1}{2} (S - S'')}{\cos \frac{1}{2} (S + S'')} . \cot \frac{1}{2} (A + A'')$ (Art. 244.)

Such values as make an angle, or a side, either negative, or not less less than 180°; and such as place the greater side opposite to the less angle, or the greater angle opposite to the less side, must (Art. 41. 72. 129.) be rejected.

The data may be such as to render this case (Art. 140. 141.) *impossible*. It may, therefore, as hath been shewn, admit of two solutions, of only one solution, or of none.

Case 6.

Let any two angles, A'', A, and a side opposite to one of them, as S'', be given, to find the other parts of the triangle.

Then may the side, S, opposite to the other given angle, A, be found by means of Art. 234. IX. Thus

$\sin S = \frac{\sin A}{\sin A'} . \sin S''$. But it will be uncertain, whether the value, thus obtained, be the value of S, or of its supplement. The remaining side, and angle, may be found, by a deduction from Naper's Analogies; as in the method last used in solving the fifth case.

Or, let ABC* be the proposed triangle; let the angle $ABC = A'$; and let A' be first sought. From the point B, describe the arch BD at right angles to the side AC (S'); and let the angle $ABD = x$.

Then (Art. 232. IV.) $\cot x = \cos S'' \tan A$,

and (Art. 238. 4.) $\sin (A' - x) = \frac{\cos A''}{\cos A} \sin x$.

Whence $A' - x$ may be said to be known; and $A' = (A' - x) + x$.

It is still, however, uncertain, whether the value obtained for $A' - x$ indicate an *acute* or an *obtuse* angle.

By a similar process the side AC (S') might be found: but its value would be subject to the same kind of uncertainty.

In this case, as in the fifth, the data may be such as to render the problem *impossible*: and if the problem be *possible*, it may have two solutions, or it may admit of only one.

(264.) SCHOLIUM. The main object of Spherical Trigonometry is the solution of this general problem: "To determine any three of the six parts of a spherical triangle, when the three remaining parts are given:"

* See the figure in Case 5.

and that object will, evidently, have been attained, when equations have been found, which exhibit the relations between four of the proposed quantities, or their functions, taken in all the possible combinations of four quantities out of six. Now, the whole number of such combinations is $\left(\frac{6\,.\,5}{1\,.\,2}\right)$ fifteen; but yet, when they are compared with one another, there are found, amongst them, only four combinations, that are essentially different: and the four theorems, marked (I), (II), (IX), (X), furnish the necessary equations. So that the combinations, and the theorems which embrace them, admit of the following classification:

Combination 1. Three sides and an angle (I.)
2. Two sides and two angles opposite to them (IX.)
3. Two sides and two angles, not both of them opposite (X.)
4. One side and three angles (II.)

The problems, which those several combinations supply, and which are really distinct problems, may be reduced to six cases: and these cases have been *separately* discussed, in Art. 263; because, although the four theorems that have been cited, are sufficient for the solution of any proposed spherical triangle, of which any three parts are given, they do not always afford the best practical methods of solution.

When a summary is wanted, of the modes of proceeding, *generally* to be followed, in the solution of oblique-angled spherical triangles, the following table may be consulted.

	Given	
(1.)	S, S', S''.	$\sin\frac{1}{2}A=\sqrt{\dfrac{\sin(P-S')\sin(P-S'')}{\sin S\sin S''}}$; $\sin A'=\dfrac{\sin S'}{\sin S}\sin A$; $\sin A''=\dfrac{\sin S''}{\sin S}.\sin A$.
(2.)	A, A', A''.	$\sin\frac{1}{2}S=\sqrt{\dfrac{-\cos p.\cos(p-A)}{\sin A'\sin A''}}$; $\sin S'=\dfrac{\sin A'}{\sin A}\sin S$; $\sin S''=\dfrac{\sin A''}{\sin A}.\sin S$.
(3.)	S', A, S''.	$\left\{\begin{array}{l}\tan\frac{1}{2}(A'+A'')=\dfrac{\cos\frac{1}{2}(S'\sim S'')}{\cos\frac{1}{2}(S'+S'')}\cot\frac{1}{2}A\\ \tan\frac{1}{2}(A'\sim A'')=\dfrac{\sin\frac{1}{2}(S'\sim S'')}{\sin\frac{1}{2}(S'+S'')}\cot\frac{1}{2}A\end{array}\right\}$; $\sin S=\dfrac{\sin A}{\sin A'}.\sin S'$.
(4.)	A', S, A''.	$\left\{\begin{array}{l}\tan\frac{1}{2}(S'+S'')=\dfrac{\cos\frac{1}{2}(A'\sim A'')}{\cos\frac{1}{2}(A'+A'')}\tan\frac{1}{2}S\\ \tan\frac{1}{2}(S'\sim S'')=\dfrac{\sin\frac{1}{2}(A'\sim A'')}{\sin\frac{1}{2}(A'+A'')}\tan\frac{1}{2}S\end{array}\right\}$; $\sin A=\dfrac{\sin S}{\sin S'}.\sin A'$.
(5.)	S, A'', S''.	$\sin A=\dfrac{\sin S}{\sin S''}.\sin A''$; $\left\{\begin{array}{l}\tan\frac{1}{2}S'=\dfrac{\cos\frac{1}{2}(A+A'')}{\cos\frac{1}{2}(A\sim A'')}\tan\frac{1}{2}(S+S'').\\ \tan\frac{1}{2}A'=\dfrac{\cos\frac{1}{2}(S\sim S'')}{\cos\frac{1}{2}(S+S'')}\cot\frac{1}{2}(A+A'').\end{array}\right\}$
(6.)	A, S, A''.	$\sin S''=\dfrac{\sin A''}{\sin A}.\sin S$;

Three rules have been already referred to, for removing the ambiguity, incident to the solution of some of the cases of oblique-angled spherical triangles: viz.

1. *The greater side is opposite to the greater angle.*
2. *The measure of any side, or angle, is less than* 180°, *or, the measure of the half of any side, or angle, is less than* 90°.
3. *No side, nor any angle, can have a negative value.*

It is evident, however, that these rules can only be applied, *after* the calculation of the unknown parts of the triangle: but if, in the fifth case, S' be the side opposite to the given angle; and if, in the sixth case, A' be the angle opposite to the given side; it has been proved (Art. 144.) that whenever

S' *is greater than* S, *and less than* $180° - S$,

or, whenever

A' *is greater than* A, *and less than* $180° - A$,

then, the unknown part is of the same species as the given part opposite to it: and this rule may be applied to determine the species of the parts of the proposed triangle, previously to any calculation having been entered into.

Thus, in either of the two ambiguous cases, after two sides and the two angles opposite to them have been determined, if the solution be carrried on, by drawing an arch from one of the angles, at right angles to the opposite side, it will be known, from Art. 134, whether that arch fall within, or without, the triangle. In the former case, the unknown quantity will be the *sum*, and, in the latter case, it will be the *difference*, of the resulting segments.

Further, when an astronomical problem is reduced to the solution of a spherical triangle, the *circumstances* of the problem will, of themselves, sometimes, exclude one of the two values found for an unknown part. In computing, for example, the zenith distance of a star, from data obtained by actual observation, if two values be found, one *less*, and the other *greater* than 90°, the latter value is, manifestly, excluded; because, if the zenith distance had really exceeded 90°, the star would have been invisible, and the observation upon it could not have been made.

PART II.

THE ELEMENTS OF

Spherical Trigonometry.

SECTION IV.

ON THE COMPUTATION OF SPHERICAL SURFACES.

Prop. I.

(265.) *Problem.* To express the surface of a spherical triangle, in terms of its three angles, and of the radius of the sphere.

Put r for the sphere's radius; π for the circumference of a circle, of which the diameter is *unity:* then (Archim. or Legendre Geom. and E. 2. 12.) the surface of the hemisphere is expressed by $2r^2\pi$; and, if T be put for the surface of the spherical triangle, of which the angles are A, A', A'',

$$T : 2r^2\pi :: (A + A' + A'') - 180^\circ : 360^\circ \text{ (Art. 222.)}$$

$$\therefore\ T = \frac{2r^2\pi}{360} \cdot (A + A' + A'' - 180^\circ)$$

$$= \frac{r^2\pi}{180}(A+A'+A''-180^0)$$

$$= \sin 1''.r^2.(A+A'+A''-180^0)^*$$

Prop. II.

(266.) *Problem.* To express the surface of a spherical triangle, in terms of any two of its sides and of the included angle.

If $\sin 1''.r^2$ be taken for an unit, then, the same notation being used as in the preceding article, and S, S', S'', being put for the sides of the spherical triangle,

$$T = A+A'+A''-180 \text{ (Art. 265.)}$$

$$\therefore -\cot \tfrac{1}{2} T = \tan \tfrac{1}{2}(A+A'+A'') \text{ (Introd. 13. 19.)}$$

$$= \frac{\tan \frac{1}{2} A + \tan \frac{1}{2}(A'+A'')}{1-\tan \frac{1}{2} A \tan \frac{1}{2}(A'+A'')} \text{ (Introd. 29.)}$$

$$= \frac{\tan \frac{1}{2} A \cos \frac{1}{2}(S'+S'') + \cot \frac{1}{2} A \cos \frac{1}{2}(S'-S'')}{\cos \frac{1}{2}(S'+S'') - \cos \frac{1}{2}(S'-S'')}$$

(Art. 244.)

$$\therefore \cot \tfrac{1}{2} T = \frac{\cos \frac{1}{2} S' \cos \frac{1}{2} S'' + \sin \frac{1}{2} S' \sin \frac{1}{2} S'' \cos A}{\sin \frac{1}{2} S' \sin \frac{1}{2} S'' \sin A}$$

(Introd. 29.)

$$= \frac{\cot \frac{1}{2} S' \cot \frac{1}{2} S'' + \cos A}{\sin A} \text{ (Introd. 17.)}$$

(267.) Cor. Hence, two spherical triangles are equal to one another, if their vertical angles be equal, and if the tangents of the halves of the sides containing the equal angles, be reciprocally proportionals. And, by

* By consulting a book of logarithms, it will be found that the logarithm of sin 1″ is, also, the logarithm of $\frac{\pi}{180}$.

means of this property, on a given sphere a spherical triangle may be described, on a given arch, that shall be equal to a given triangle, of the sphere, and shall have one of its angles equal to the angle of the given triangle. Also, an isosceles spherical triangle may be described, which shall be equal to a given triangle, and shall have its vertical angle, equal to the vertical angle of that given triangle.

Prop. III.

(268.) *Problem.* To express the surface of a spherical triangle, in terms of its sides.

Let the same notation be used as in Art. 265. 266. and let P be put for the semi-sum of the three sides of the triangle.

Then, $1+\cos A = \dfrac{2 \sin P \sin (P-S)}{\sin S' \sin S''}$ (Art. 263. Case 1.)

And, $1-\cos A = \dfrac{2 \sin (P-S') \sin (P-S'')}{\sin S' \sin S''}$.

If, therefore, the square root be taken of the product of the two last equations,

$$\sin A = \frac{2 \sqrt{[\sin P \sin (P-S) \sin (P-S') \sin (P-S'')]}}{\sin S' \sin S''}$$

(Introd. 15.)

Again, since $\cos A = \dfrac{\cos S - \cos S' \cos S''}{\sin S' \sin S''}$ (Art. 230.

and that $\cot \frac{1}{2} S' \cot \frac{1}{2} S'' = \dfrac{1+\cos S'}{\sin S'} \cdot \dfrac{1+\cos S''}{\sin S''}$

(Introd. 30.)

$\therefore \cot \frac{1}{2} S' \cot \frac{1}{2} S'' + \cos A = \dfrac{1+\cos S + \cos S' + \cos S''}{\sin S' \sin S''}$;

$$\therefore \cot. \tfrac{1}{2}\, T = \frac{\cot \frac{1}{2} S' \cot \frac{1}{2} S'' + \cos A}{\sin A} \quad \text{(Art. 266.)}$$

$$= \frac{1 + \cos S + \cos S' + \cos S''}{2\sqrt{[\sin P \sin (P - S) \sin (P - S') \sin (P - S'')]}}.$$

(269.) COR. If the spherical triangle (T) be equilateral,

$$\cot \tfrac{1}{2}\, T = \frac{1 + 3 \cos S}{2 \sqrt{\sin \left(\frac{3S}{2}\right) \sin^6 \left(\frac{S}{2}\right)}}.$$

(270.) SCHOLIUM. The following remarkable expression for the surface (T) of a spherical triangle, which may easily be deduced from what has been premised, was first given by Lhuillier;

$$\tan \tfrac{1}{2}\, T = \sqrt{\left(\tan \frac{P}{2} . \tan \frac{P-S}{2} . \tan \frac{P-S'}{2} . \tan \frac{P-S''}{2}\right)}.$$

This form is manifestly analogous to the equation which exhibits the surface of a plane rectilineal triangle, in terms of its sides, and which might be deduced from the spherical form, by supposing the sphere's radius to be infinitely great. It serves, also, to shew, that when the sides of a spherical triangle are very small, compared with the radius of the sphere, its surface is nearly equal to that of a plane rectilineal triangle, contained by sides that are equal, in length, to the sides of the spherical triangle, each to each. For (Introd. 35.) a circular arch, when it is very small, in comparison of the radius, may be taken for its tangent, without any considerable error: and, in that case, Lhuillier's equation becomes

$$\tfrac{1}{4}\,T = \tfrac{1}{4}\sqrt{[P\,.\,(P-S)\,.\,(P-S')\,.\,(P-S'')]},$$

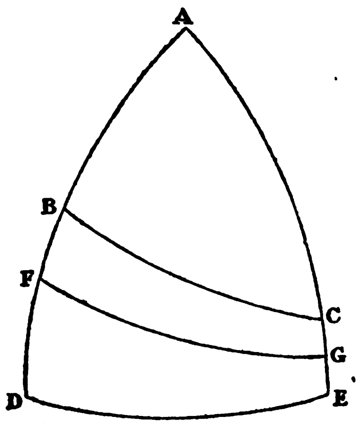
$$\text{or } T = \sqrt{[P\ (P-S').(P-S').(P-S'')]};$$

which also (Introd. 34.) expresses the surface of a plane rectilineal triangle, of which the sides are S, S' and S''.

Prop. IV.

(271.) ***Problem.*** Two sides of a quadrilateral spherical figure being at right angles to its base, to express in terms of those two sides, and of the base, the surface of the figure; all its sides being arches of great circles of the sphere.

Let the two sides *BD*, *CE*, of the quadrilateral spherical figure *BDEC*, which is bounded by arches of great circles of the sphere, be at right angles to the base *DE*:



It is required to express the surface of the figure *BDEC*, in terms of the arches *BD*, *CE* and *DE*.

Produce *BD* and *EC*, until they meet in *A*: then (Art. 51. 36.) *DA* and *EA* are quadrants; and (Art. 54.) the side *DE* is the measure of the spherical angle *A*.

But (Art. 265.) if $\left(\frac{r^2\pi}{180}\right)$ be taken as an unit, and if Q be put for the figure $BDEC$,

$$\triangle\, ADE = \angle\, A$$

$$\triangle\, ABC = \angle\, A + \angle\, B + \angle\, C - 180;$$

$$\therefore\ \triangle\, ADE - \triangle\, ABC = Q = \angle\, B + \angle\, C - 180;$$

$$\therefore\ \tfrac{1}{2}Q = \tfrac{1}{2}(B+C) - 90;$$

$$\therefore\ \tan\tfrac{1}{2}Q = \cot\tfrac{1}{2}(B+C)$$

$$= \frac{\cos\frac{1}{2}(AC+AB)}{\cos\frac{1}{2}(AC-AB)} \cdot \tan\tfrac{1}{2}A \quad \text{(Art. 244.)}$$

$$= \frac{\sin\frac{1}{2}(BD+CE)}{\sin\frac{1}{2}(BD-CE)} \cdot \tan\tfrac{1}{2}A \quad \text{(Introd. 13.)}$$

If, then, B be put for the base, and H, H', for the two sides that are at right angles to it,

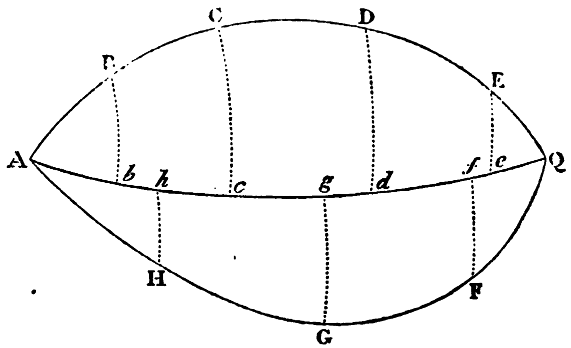
$$\tan\tfrac{1}{2}Q = \frac{\sin\frac{1}{2}(H+H')}{\cos\frac{1}{2}(H-H')} \cdot \tan\tfrac{1}{2}B.$$

(272.) Scholium. The problem, which has been solved in Art. 271, may often be advantageously employed in determining the number of square miles contained in any large tract of the Earth's surface. For, when the latitudes, and the longitudes, of several points, in the boundary of that region, are known, the whole space may be very conveniently divided into quadrilateral spherical figures, having, each of them, two sides perpendicular to the base; and the calculation may then be carried on with great facility.

Thus, let $BFGC$ be the space to be computed; let the points B, F be in the circle of latitude ABD; and

the points *C*, *G* in the circle of latitude *ACE*; let *DE* be the equator, and *A* its pole. The angle *A*, or the arch *DE*, measures the difference of the longitudes of *B* and *C*, or of *F* and *G*; also *BD*, *FD*, *CE*, and *GE*, are the latitudes of the points *B*, *F*, *C*, and *G*, respectively: wherefore, by means of the equation, found in Art. 271, the spaces *BDEC* and *FDEG* may be computed; and their difference, *BFGC*, is the space required.

Again, let *ABCDEQFGH* be the tract to be mea-



sured; let it lie on both sides of the equator *AQ*; and let the latitudes, and the longitudes, of the several points *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*, be given: Then, if circles of latitude, *Bb*, *Cc*, *Dd*, &c. be supposed to be drawn through those several given points, and if the points themselves be supposed to be joined, by arches of great circles, the whole space will be divided into four right-angled spherical triangles, and five such quadrilateral figures, as are described in Art. 271: its superficial content may, therefore, readily be found, by means of that and of the preceding Articles.

PROP. V.

(273.) ***Problem.*** To bisect an isosceles quadrantal triangle, by an arch of a great circle, drawn through a given point, in one of its equal sides.

Let ADE* be an isosceles quadrantal triangle, and B a given point in AD, one of its equal sides: It is required to bisect the triangle ADE, by an arch of a great circle, drawn through B.

Let the arch BC be supposed to bisect the given triangle; and, the same notation being employed, as in Art. 271, let x be put for the arch CE: Then, (Art. 271.)

$$\frac{\sin \frac{1}{2}(H+x)}{\cos \frac{1}{2}(H-x)} = \frac{\tan \frac{1}{4} B}{\tan \frac{1}{2} B} = \frac{1}{b},$$

b being put for $\dfrac{\tan \frac{1}{2} B}{\tan \frac{1}{4} B}$;

$$\therefore \frac{\sin \frac{1}{2} H \cos \frac{1}{2} x + \cos \frac{1}{2} H \sin \frac{1}{2} x}{\cos \frac{1}{2} H \cos \frac{1}{2} x + \sin \frac{1}{2} H \sin \frac{1}{2} x} = \frac{1}{b}$$

(Introd. 27, 28.)

$$\therefore \frac{\tan \frac{1}{2} H + \tan \frac{1}{2} x}{1 + \tan \frac{1}{2} H \tan \frac{1}{2} x} = \frac{1}{b} \text{ (Introd. 17.)}$$

$$\therefore b . \tan \tfrac{1}{2} H + b \tan \tfrac{1}{2} x = 1 + \tan \tfrac{1}{2} H \tan \tfrac{1}{2} x;$$

$$\therefore \tan \tfrac{1}{2} x = \frac{1 - b \tan \frac{1}{2} H}{b - \tan \frac{1}{2} H}:$$

whence, the value of x, or CE, becomes known; and the arch of a great circle, which joins B, C, bisects the given isosceles quadrantal triangle ADE.

* See the Figure in Art. 271.

PART II.

THE ELEMENTS OF

Spherical Trigonometry.

SECTION V.

ON THE FORMATION OF A TECHNICAL MEMORY, FOR THE PURPOSES OF SPHERICAL TRIGONOMETRY.

(274.) In attaining a knowledge of the theory of Spherical Trigonometry, no greater exercise of the memory is required, than that which is necessarily implied in the mental process, of following the connexion of the chain of proofs. When, however, the *study* of Trigonometry is finished, and occasion is, afterwards, found for a practical application of its theorems, it is plain, that, unless the theorems themselves can be recollected, reference must continually be made, to the books in which they are exhibited.

The objections which may be urged, against the

latter mode of supplying an immediate want of such theorems, are very obvious. It is not only toublesome, but, under some circumstances, it may become impracticable; and it always consumes much more time than does the mere act of recollection.

On the other hand, although the memory, in some individuals, appears to be naturally strong, and although it is, perhaps, of all our faculties the most manageable, and the most susceptible of improvement, yet there are very few persons, who would chuse to burden it with the details of Spherical Trigonometry. It becomes, therefore, necessary to enquire, whether there are any methods, by which the memory may well be relieved from such a weight of matter, uninteresting in itself, and chiefly valuable on account of the purposes to which it may be applied.

(275.) Now, there are, in reality, three ways, by which this kind of relief may be afforded.

The first consists merely in directing the attention to those particular theorems, which are of the greatest importance, and of the most frequent use; thus abridging the *quantity*, without altering the *form*, of what is to be remembered.

The second mode of relief is the invention of general rules; which, although they are comprised in few words, and are, therefore, easily gotten by heart, do nevertheless,

comprehend many particular cases, which it would be irksome to remember, as so many distinct propositions.

The third method is founded on the well-known principle of the association of ideas; and consists in the substitution of such mathematical forms as are easily retained; and which, being recollected, suggest to the mind more complicated expressions.

It is evident, that only the two last of these methods can, with propriety, be said to belong to the province of Mnemonics. We shall proceed, however, to a brief illustration of all of them.

(276.) The most extensively useful, of all spherical theorems, is that which foreign writers have called *the Theorem of the Four Sines*; according to which, the sines of the sides, are proportional to the sines of the opposite angles, of a spherical triangle: and it is that, of all others, which is the most easily retained.

Of next, if not of equal importance, are Naper's four Analogies, demonstrated in Arts. 244. 246. And it appears, from the table, exhibited in Art. 264, that in addition to those analogies, and the theorem of the four sines, only two other expressions, one for the sine of half an angle, the other for the sine of half a side, are wanted, for the ready solution of all the cases of oblique-angled spherical triangles; which cases necessarily include those, also, of right-angled spherical triangles:

and the expressions, which have been indicated, are all of them adapted to the best mode of computation, namely, that of logarithms.

It would not, then, be very unreasonable to exact, from the student, so much labour as is required, to fix in the memory the substance of seven different theorems; of which the first, in the order here assigned to them, can hardly ever be forgotten; the second and the third have not only a strong resemblance to one another, but a close connexion with the fourth, and the fifth; so that the effort of the memory is chiefly confined to the perfect retention of the first analogy: and the sixth of the requisite theorems may, likewise, in some measure, serve to recall the seventh.

The three fundamental equations,

$$\cos S = \cos S' \cos S'' + \sin S' \sin S'' \cos A;$$
$$\cos A = \sin A' \sin A'' \cos S - \cos A' \cos A'',$$
$$\cos S \cos A' = \cot S'' \sin S - \sin A' \cot A'',$$

deserve, also, to be committed to memory, although the solution of spherical triangles may be effected without them.

The forms, which have been recommended to be used, in general, for the solution of oblique-angled spherical triangles, are not, it must be acknowledged, the best that can be employed in solving right-angled triangles. It happens, however, that the second mode of

relieving the memory comes specially in aid of this defect.

(277.) The six theorems, which are best adapted to the solution of right-angled spherical triangles, are all comprehended in two rules, invented by Naper, and afterwards explained and improved by Gellibrand and Manduit. Calling the hypotenuse, the two angles adjacent to it, and the complements of the two remaining sides, the *circular parts*, taking any one of these as the *middle part*, and calling those two of the circular parts, which are adjacent to it, the *adjacent extremes*, and the two remaining circular parts, the *separate extremes*, then, the tabular radius being unity,

"The *cosine* of the *middle part* is equal, first, to the rectangle contained by the *sines* of the *separate extremes*; and secondly, to the rectangle contained by the *co-tangents* of the *adjacent extremes*."

Or, if a part which is unknown, but which is not required to be found, be taken as the *middle part*, the two rules may be reduced to this one: "The rectangle contained by the *co-tangents of the adjacent extremes* is equal to the rectangle contained by the *sines of the separate extremes*.

Thus, if A be put for the right angle, and S for the hypotenuse, then, beginning with the oblique angle A' at the foot of the hypotenuse, the circular parts are

$$A', S, A'', (S' \sim 90), (S'' \sim 90).$$

And it is manifest that, in any combination of three, out

of the five circular parts, two of the parts will either be *adjacent* to, or *separate* from the remaining part, A' and $(S''{\sim}90)$ being supposed to adjoin to one another*. If, therefore, the rules be true, and any two of the parts be given, the rest may be found: because the rules will always furnish three independent equations, in each of which one of the three parts, required, is the only unknown quantity.

It is unnecessary to enter upon any direct investigation of these rules: because when all the results, which they produce, are expressed at length, they are found to be merely the enunciations of six theorems, which have already been demonstrated. If, however, Forms V. and VII. (Art. 232.) be successively applied to any assumed right-angled spherical triangle, and to its two complemental triangles, it will be made evident, that the rules are true †.

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* If the space between two concentric circles be divided into five compartments, and a *Circular Part*, as it has been called, be placed in each, in the order in which they are set down in the text, then A' and $(S''{\sim}90)$ will adjoin to another, and all the definitions, relative to the circular parts, will be clearly understood.

† Naper's rules are, in reality, nothing more than a contrivance to express in *few* words, of *extensive* signification, what is otherwise enunciated in a greater number of words of more limited signification: and the merit of the contrivance is founded on this principle,—that it is easier to retain the extended meaning of the new terms employed, and the short form of words, in which they are used, than the substance of the six theorems, which the rules comprehend.

An eminent French Astronomer has however avowed, that it has always been less irksome to him to retain the six theorems themselves, than to call to mind, and to apply, Naper's rules. There is, he thinks, an

(278.) The method of relieving the memory, which yet remains to be explained, is founded on the analogy, which exists between the two branches of Trigonometry, Plane and Spherical. The principal theorems of Plane Trigonometry have never been accounted hard to be retained, or even to be investigated anew, if they have been forgotten: and, when they have once been attentively compared with the corresponding theorems of Spherical Trigonometry, to which they bear a very striking resemblance, they may serve, even afterward, to revive these latter theorems in the memory.

Thus, if A, A', A'', be put for the angles S, S', S'', for the opposite sides, and P for the semi-perimeter, of a triangle, either plane or spherical, then,

In a plane triangle,

(1.) $$\frac{S}{S'} = \frac{\sin A}{\sin A'}; \quad \frac{S'}{S''} = \frac{\sin A'}{\sin A''}.$$

(2.) $$\frac{(P - S').(P - S'')}{S'.S''} = \sin^2 \tfrac{1}{2} A.$$

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an inconvenience, in having to substitute, for the base and for the perpendicular, their complements; in being obliged to consider which parts are *adjacent* to, and which are *separate* from, the middle part; and lastly, in having the quantity which is sought, sometimes a *middle* part, and sometimes one of the *extremes:* and he asserts, that an attention to these details consumes more time than the calculation itself. Certainly, neither in his opinion, nor in his experience, is he altogether singular. It may, nevertheless, be doubted, whether a person who, from constant practice, cannot fail to have the six theorems themselves fixed in his memory, be a fair judge of the value of rules, which, to him at least, must necessarily be useless.

$$(3.)\quad \frac{S + S'}{S \sim S'} = \frac{\tan\frac{1}{2}(A + A')}{\tan\frac{1}{2}(A \sim A)} \cdot \left(\frac{\frac{1}{2}S''}{\frac{1}{2}S''}\right).$$

In a spherical triangle,

$$(1.)\quad \frac{\sin S}{\sin S'} = \frac{\sin A}{\sin A'};\ \frac{\sin S'}{\sin S''} = \frac{\sin A'}{\sin A''}.$$

$$(2.)\quad \frac{\sin(P - S') \cdot \sin(P - S'')}{\sin S' \sin S''} = \sin^2 \tfrac{1}{2} A.$$

$$(3.)\quad \frac{\tan\frac{1}{2}(S+S')}{\tan\frac{1}{2}(S\sim S')} = \frac{\tan\frac{1}{2}(A+A')}{\tan\frac{1}{2}(A\sim A')} \cdot \left(\frac{\tan\frac{1}{2}S''}{\tan\frac{1}{2}S''}\right).$$

Or, (Introd. Art. 17.)

$$\frac{\tan\frac{1}{2}(S + S')}{\tan\frac{1}{2}(S \sim S')} = \frac{\dfrac{\cos\frac{1}{2}(A \sim A')}{\cos\frac{1}{2}(A + A')} \cdot \tan\frac{1}{2}S''}{\dfrac{\sin\frac{1}{2}(A \sim A'')}{\sin\frac{1}{2}(A + A')} \cdot \tan\frac{1}{2}S''}.$$

Where it is remarkable, that the very same kind of trigonometrical functions of the *angles* obtain, in both branches; and that, in the spherical forms, the functions of the *sides* are of the same kind as the functions of the *angles*. The last spherical form is easily deduced from that which precedes it; and by equating the two numerators, and also the two denominators, there result two of Naper's four analogies; and the two analogies thus obtained may readily be translated into the remaining two, by means of the polar triangle. If the second spherical form be, likewise, applied to the parts of the polar triangle, there will result the only additional form which is wanted, for the solution of oblique-angled spherical triangles; namely,

$$\sin^2 \tfrac{1}{2} S = \frac{-\cos p \, . \cos (p - A)}{\sin A' \sin A''},$$

where p is put for the semi-sum of the angles of the triangle.

But, if one of the angles, as A, be a right angle, then,

In a plane triangle,	In a spherical triangle,
(1.) $\frac{S'}{S} = \sin A'$.	(1.) $\frac{\sin S'}{\sin S} = \sin A'$.
(2.) $\frac{S''}{S} = \cos A'$.	(2.) $\frac{\tan S''}{\tan S} = \cos A$.

and, by applying these two spherical forms to the parts of the complemental triangle, there result the remaining four theorems, which are used in solving right-angled spherical triangles.

Again, the solution of oblique-angled triangles, it is well known, may be reduced to that of right-angled triangles.

Let D and D' denote the segments of the base, V and V' the segments of the vertical angle A'', of a plane or spherical triangle, made by a perpendicular let fall from A'', on S''; so that D is adjacent to the angle A, and opposite to the segment V; then,

In a plane triangle,	In a spherical triangle.
$\frac{S''}{S + S'} = \frac{S \sim S'}{D \sim D'}$.	$\frac{\tan \frac{1}{2} S''}{\tan \frac{1}{2} (S + S')} = \frac{\tan \frac{1}{2} (S \sim S')}{\tan \frac{1}{2} (D \sim D')}$.

Whence, by the properties of the polar triangle,

$$\frac{\cot \tfrac{1}{2} A''}{\tan \tfrac{1}{2} (A + A')} = \frac{\tan \tfrac{1}{2} (A \sim A')}{\tan \tfrac{1}{2} (V \sim V')}.$$

and these two spherical forms, together with those which solve right-angled spherical triangles, are sufficient for the solution of all the cases of oblique-angled spherical triangles.

A recollection, therefore, of the fundamental principles of Plane Trigonometry, and of the very striking analogy which has been pointed out, together with the well-known properties of the polar, and the complemental triangles, may serve to re-produce all that is wanted, for the solution of the most important problems of Spherical Trigonometry.

(279.) The expression for the surface of a spherical triangle in terms of its three angles and the radius of the sphere, is too simple and too remarkable, to need, at all, the aid of an artificial memory: and, when the three angles are not given, they may be found, if the data be sufficient, by a solution of the triangle.

(280.) Upon the whole, the last of the three methods of relieving the memory, ought, perhaps, chiefly to be recommended to those who are new to this subject, and who are not likely to have very frequent need to employ the principal theorems of Spherical Trigonometry: they will thus easily recover the requisite forms, whenever an occasion calls for them. And, if these forms be often wanted, they will, undoubtedly, be fixed in the memory, by repeated use; so that the aid of any indirect method of arriving at them, will become altogether unnecessary.

PART II.

THE ELEMENTS OF

Spherical Trigonometry.

SECTION VI.

ON THE SOLUTION OF SPHERICAL TRIANGLES BY GEOMETRICAL CONSTRUCTIONS.

PROP. I.

(281.) *Problem.* THE numerical values of any three parts of a proposed spherical triangle being given, to find the remaining three parts, by geometrical constructions, made on the surface of a given sphere.

CASE 1.

Let the values of the three sides of the triangle be given, to find the three angles.

On the given sphere, describe (Art. 68.) three arches of great circles; and, by means of a compass and a scale

of chords, cut off from them parts (Introd. 3.) equal in value to the three given sides of the triangle, each to each: then, (Art. 94.) make on the sphere's surface, a triangle, the sides of which shall be equal to the three arches so described, each to each: describe also (Art. 76.) the polar triangle; and the measures of its sides, taken by the compass and the scale of chords, will be (Art. 78.) the supplements of the required angles: whence the angles themselves will become known.

Case 2.

Let the values of the three angles be given, to find the three sides of the triangle.

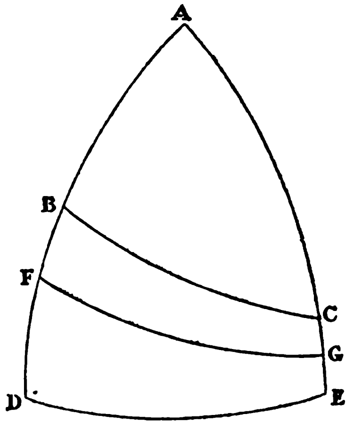
Describe, as in the first case, a triangle on the sphere's surface, having its three sides, each the measure of the supplement of one of the given angles; then the measures of the sides of its polar triangle, taken by a compass and a scale of chords, will, manifestly, give the values of the three sides, which are sought.

Case 3.

Let the values of two sides, and of the included angle be given, to find the other parts of the triangle.

On the given sphere, describe, as in the first case, an arch, *AD* of a great circle; from either of its extremities, *A*, as a pole, at the distance of a quadrant, describe the great circle *DE*; and make, by means of

the compass and the scale of chords, the arch *DE* equal to the measure of the given angle; join *A*, *E*; and from



AD and *AE*, produced if necessary, cut off, (Introd. 3.) *AB* and *AC* equal in value to the two given sides of the triangle, each to each; join (Art. 66,) *B*, *C*: the sides *BC* may then be measured, by means of the same instruments; as may, likewise, the angles *B* and *C*, if from each of those angular points, as a pole, a great circle be described, cutting the two sides, which contain the angle to be measured.

Case 4.

Let there be given the values of any one of the sides, and of the two angles adjacent to it, to find the other parts of the triangle.

Describe an arch of a great circle of the sphere, and make it equal, in value, to the given side, as in the former cases: make also, as in the third case, at each extremity

of the arch so described an angle equal in value to one of the given angles: and the remaining angle, of the resulting spherical triangle, and its two other sides, being measured, as before, will, manifestly, give the values of the parts which are sought.

CASE 5.

Let the values of two angles, and of a side opposite to one of them, be given, to find the other parts of the triangle.

Make, as in the preceding cases, an arch, of a great circle, equal in value to the given side, and, from either of its extremities, draw another such an arch, making with it, as in the third case, a spherical angle equal to the given angle that is adjacent to the given side: Make, also, in the same manner, another angle, on the sphere's surface, equal to the other given angle: then (Art. 117.) from the other extremity of the arch, which was made equal, in value, to the given side, draw a third arch of a great circle, making with the other an angle, equal to the angle last constructed: and the remaining angle, and the two other sides, of the resulting triangle, being measured by means of the compass and the scale of chords, will give the values of the parts which were required to be found; with the same degree of ambiguity, however, as when the problem is solved by numerical calculation.

CASE 6.

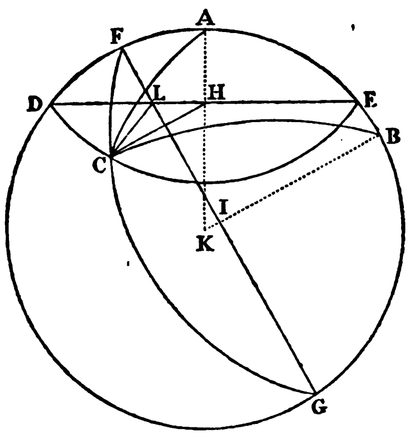
Let there be given the values of any two sides, and of an angle opposite to one of them, to find the other parts of the triangle.

On the given sphere, construct, by the help of the fifth case, a spherical triangle, having two of its angles equal to the supplements of the given sides, and having a side opposite to one of them equal in value to the supplement of the given angle: describe, also, its polar triangle: then (Art. 78.) will the unknown parts of the polar triangle, when they are measured, as before, give the values of the parts that are sought.

Prop. II.

(282.) *Problem.* The numerical values of any three of the six parts of a spherical triangle being given, to find the three remaining parts, by geometrical constructions made on a plane.

Let it be supposed, that *ABC* is a spherical triangle, on any given sphere, similar to the proposed triangle:



from *A* and *B*, as poles, at the distances *AC* and *BC*, let there be described (Art. 59.) the circles *DCE, FCG,*

meeting the circumference of the great circle *AB*, in the points *D* and *E*, *F* and *G*, respectively. Let *K* be the sphere's center; and suppose the points *K* and *A*, *K* and *B*, *D* and *E*, *F* and *G*, to be joined by straight lines. Then (Art. 50.) the great circle *ABGD* is at right angles to each of the two lesser circles *DCE*, *FCG*; and it passes through both their diameters, *DE* and *FG*: wherefore, *DE* and *FG* are in the same plane, and cut each other; let them cut one another in *L*. Again (Art. 22.) *AK* passes through the center *H*, of the circle *DCB*, and *KB* passes through the center *I* of the circle *FCG*: let *L*, *C*, and *H*, *C*, be joined: and, because the circles *DCE*, *FCG* are, both of them, perpendicular to the circle *ABGD*, their common section, *CL*, is (E. 19. 11.) perpendicular to the plane of *ABGD*, and, therefore, also perpendicular (E. Def. 3. 11.) to the diameters *DE* and *FG*.

Now, the arch *CE* measures (Art. 54.) the spherical angle *CAE*; and (E. 33. 6.) it also measures the plane angle *CHE*; so that the plane angle *CHE* is equal to the spherical angle *CAE*: and, in the same manner it may be shewn, that the plane angle, subtended by the arch *FC*, at the center *I*, is equal to the spherical angle *ABC*. It is evident, also, that *LC* (Introd. 6. 20.) is the sine, in the circle *DCE*, of the angle *CHE*, i. e. of the spherical angle *CAE*; and that, in the circle *FCG*, it is, in like manner, the sine of the spherical angle *ABC*.

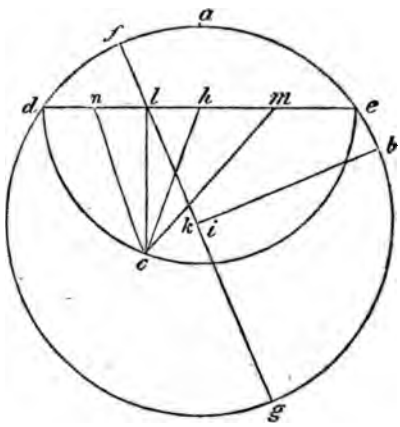
From what has been thus premised, the following

constructions, which furnish the solution of the proposed problem, in all its cases, may manifestly be derived.

Case 1.

Let the three sides *AB*, *AC*, and *BC* of a spherical triangle be given, to find the three angles.

From any assumed point *k*, as a center, and at any distance from it, describe a circle *adge*; from any point



a in its circumference, set off, by means of the scale of chords, the arches *ad* and *ae*, each of them equal in value to the side *AC*, and the arch *ab* equal to the side *AB*; also, from *b*, set off the arches *bf* and *bg*, each of them equal to the side *BC*: join *d*, *e* and *f*, *g*; and let *de* be cut by *fg* in *l*; upon *de*, as a diameter, describe the semi-circle *dce*; from *l* draw *lc* perpendicular to *de*; find the center *h* of the circle *dce*, and join *h*, *c*. Then, from what has been premised, it is manifest, that the angle *che* is equal to the spherical angle *A*, contained by

the sides AB and AC. In the same manner, may a plane rectilineal angle be found, which shall be equal to the angle contained by any other two of the three given sides of the triangle: and the angles, thus found, may be measured (Introd. 3.) by a scale of chords.

Or, if fg be bisected in i, if from c, as a center, at a distance equal to ig, a circle be described cutting de in m, and if c, m be joined, the angle hmc will be equal to the spherical angle B, contained by the two sides AB and BC.

For, $cm : ch :: \sin \angle chm : \sin \angle cmh$ (Introd. 21.) But cm, being equal to ig, is the sine of bg, or of BC; and ch, being equal to he, is the sine of ae, or of AC; wherefore (Art. 233.)

$$\sin \angle A : \sin \angle B :: \sin \angle chm : \sin \angle cmh.$$

But the angle chm has been shewn to be equal in value to the spherical angle A; therefore (E. 14. 5.) the spherical angle B, and the angle cmh, have equal sines; and the angle B is consequently equal either to the angle cmh or cme: and the species of the angle B being known, from what has been proved in Sect. 4. Part I. it will, therefore, be known whether that angle be equal to the angle cmh, or to cme.

Case 2.

Let the three angles A, B, and C, of a spherical triangle be given, to find the three sides.

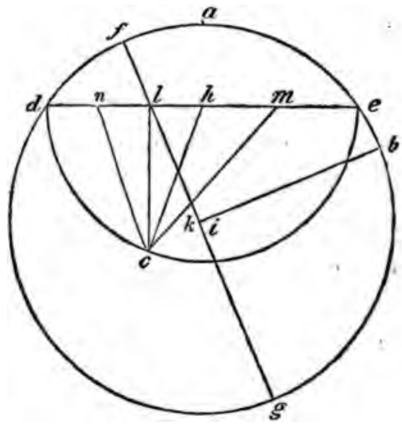
It is manifest, from Art. 78, that if, by the first case, the three angles be found of a spherical triangle, having

its three given sides equal in value to the supplements of *A*, *B*, and *C*, each to each, the angle, so found, will be equal in value to the three sides of the proposed triangle: for the one triangle will be the polar triangle of the other.

Case 3.

Let any two sides of a spherical triangle *ABC*, as *AC* and *AB*, and the included angle *A*, be given, to find the remaining parts.

Describe, as before, about any point *k*, any circle *adge*; and make the arches *ae* and *ad* each equal to the given



side *AC*; make, likewise, *ab* equal to the other given side *AB*; join *d*, *e*; upon *de* as a diameter, describe the semicircle *dce*; find the center *h*, of that circle; at the point *h*, in *ch*, make the angle *ehc* equal to the given spherical angle *A*; from *c* draw *cl* perpendicular to *de*; join *b*, *k*, and through *l* draw the straight line *flig* perpen-

dicular to *bk*, cutting *bk* in *i*, and meeting the circumference of the circle *adb* in *f* and *g*: then is the arch *bg*, or *bf*, manifestly equal in value to the side *BC*, of the proposed triangle, which was required to be found.

Further, if *cm* be made, as before, equal to *fi*, or *ig*, the half of *fg*, then will the angle *cmh* be equal to one of the spherical angles *B*, which was to be found. And, in the same manner, may the remaining angle be determined.

Case 4.

Let any one side, and the two angles adjacent to it, of a spherical triangle, be given, to find the remaining parts.

Suppose two sides of another spherical triangle, equal, in value, each to the supplement of one of the given angles, to contain an angle equal in value to the supplement of the given side; and find the unknown parts of this suppositious triangle, by the third case: the parts so found will (Art. 78.) be equal in value to the parts of the proposed triangle, that were sought.

Case 5.

In a spherical triangle *ABC*, let there be given any one side, as *AC*, and two angles *A* and *B*, of which the angle *B* is opposite to *AC*, to find the remaining parts.

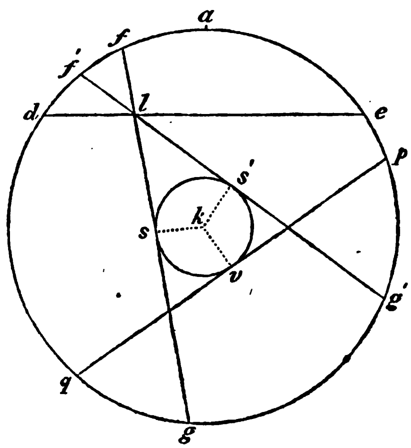
Describe, as before, any circle *ade* *; make the arches *ad* and *ae* each equal in value to the given side *AC*;

* See the Figure in Case 3.

join *d, e*; upon *de* as a diameter describe a semi-circle *dce*, and let *h* be its center; at the point *h*, in *eh*, make (Introd. 3.) the angle *ehc* equal to the given angle *A*, and draw *cl* perpendicular to *de*. Then, it is evident, from the preliminary observations, that *cl* is the sine of the other given angle *B*, in a circle, having for its diameter *fg*; which line is yet to be determined, both in length and in position. First, then, an angle, and the actual length of its sine, are given, to find the radius of the circle, to which the sine belongs.

Make, therefore, at the point *c*, in *lc*, the angle *lcn* equal to the complement of the given angle *B*; wherefore (E. 32. 1.) the angle *cnl* is equal to the angle *B*: and *cn* is equal to the radius which was required to be found.

The problem, therefore, is now reduced to this; "Through a given point *l*, within the given circle *adge*,



" to draw a chord, which shall be equal to a given straight " line, the double of *cn*." Hence, in the given circle *ade*,

of which k is the center, place (E. 1. 4.) the straight line pq, equal to the double of cn; draw kv perpendicular to pq; from k, as a center, at the distance kv, describe a circle svs'; lastly, through l draw (E. 17. 3.) two straight lines flg, $f'lg'$, touching the circle svs' in s and s' respectively: then, since (E. 16. 3. 14. 3.) fg and $f'g'$ are, each of them, equal to pq, it is evident, that the side BC, of the proposed spherical triangle, opposite to the given angle A, is equal in value, either to the half of the arch fag, or else to the half of the arch $f'ag'$: and, upon either of these suppositions, the remaining parts of the proposed triangle may be found, by the help of the preceding cases.

Case 6.

Let any two sides of a spherical triangle, and an angle opposite to one of them be given, to find the remaining parts.

Instead of the actual data belonging to the proposed spherical triangle, assume, as data, the corresponding parts of its polar triangle: then may all the unknown parts of this latter triangle be found, by the help of the fifth case; and thence (Art. 78.) the parts of the proposed triangle, which were required to be found, will be known.

(283.) Scholium. It has been shewn that spherical triangles may be solved, either by pure calculation or by geometrical constructions; and it is easy to estimate the comparative merits of these two distinct methods.

If practical utility were to be entirely left out of the

consideration of this question, perhaps either method might as well deserve to be studied as the other; nor, upon any supposition, will either of them be totally disregarded by him who makes the mathematics his principal literary pursuit. For, it is desirable, that he should have a choice of the two methods, even when both are practicable; and he may sometimes be so circumstanced, as to be able to avail himself of the one, and not of the other. But, in reality, Trigonometry could hardly be said to be worth acquiring, if it were to be considered as a mere theory: any value, which may be fairly assigned to it, as a mental speculation, almost entirely vanishes, in comparison with the highly important uses, to which it is subservient, in Geography, in Navigation, and in Astronomy. Every one, therefore, who takes the trouble of learning Trigonometry, must be supposed to have in view its application: and, if great exactness be required, in the answers to the trigonometrical questions, which he purposes to resolve, he must always have recourse to algebraic forms and to numerical calculations. For, although he will, even thus, seldom obtain a result absolutely exact, yet (Introd. 4.) will he always be able to approach much more nearly to that limit, by such means, than he can do, by graphical constructions. When, however, so much is conceded in favour of the arithmetical method, it is supposed, not only that the tables of logarithms, which are used, have so many places of figures, as to leave little room for want of precision, on that score, but that they are perfectly free, also, from errors of the press. Now, neither of these

suppositions ought, perhaps, gratuitously to be admitted; and the latter of them can, hardly in any case, be safely made.

But, if a very close approximation to truth be not necessary to his purpose, it is not improbable, that the learner will prefer, if he be taught it, the geometrical mode of solution. Pleasure can hardly be attributed either to the act of copying out the logarithms of numbers, or to that of afterwards adding and subtracting them; and, in either of these processes, mistakes may easily be made, from hastiness, or from relaxed attention.

The tracing out of geometrical diagrams, on the contrary, is a kind of petty architecture; it furnishes an exercise neither fatiguing, nor altogether devoid of amusement; the whole series of its operations, and their results, lie within a very small space; the whole work approves itself, as it goes on, to the eye, as well as to the mind: and if the elements of the theory of Geometry have been well understood and digested, there is little danger of committing any error in its practice. In order, however, to diminish, as much as is possible, the amount of its deviations from mathematical precision, care must be taken to construct as large a figure; and, consequently, to use as large a scale of chords, as conveniently may be.

The quantities of time, also, consumed by the one, and the other of these methods of solution, have been compared: and an eminent Italian mathematician has

asserted, that in any case of spherical triangles, the answer, in degrees and minutes, may always be obtained in the course of five minutes of time. But this can only be true, when it is asserted of professional calculators, who, by long use, have become perfectly familiar with their work.

Upon the whole, they who have time and opportunity, will learn, and perhaps sometimes employ, both the methods; proving the one by the other. If, however, only one of the two methods can be acquired, that of computation is, doubtless, the more valuable. But it ought not, on the other hand, to be overlooked, that much less reading is required to become master of the geometrical method, than to comprehend the investigation of the rules, which serve for solving spherical triangles algebraically. There may, therefore, be some students, who are not likely to have occasion for the practice of Trigonometry, in any other way than that of amusement, and who, either from distaste, as to the subject itself, or from natural indolence, shrink from the labour of learning this latter mode of solution, but whom it may be very practicable to teach the other mode. On all these accounts, it should seem, that the method which has been laid down, in this last section, may well deserve to be rescued from the neglect, we might almost say, from the oblivion, into which it has fallen.

TABLES

OF

TRIGONOMETRICAL FORMS.

THE following Tables will be found very convenient, for the management of all questions which are connected with Trigonometry. They serve, indeed, as a key to many useful and to many difficult problems; and the student can hardly fail to derive some considerable aid from having them before him, whilst he is employed in acquiring a knowledge of Astronomy. The *principles* of Trigonometry it is sufficient *once* to have learnt. Its *results* are *perpetually* wanted, through the whole course of the Mathematics; and it is plain that they may the most easily be found, when they are set down in tables, apart from their demonstrations.

The selection which is here exhibited of Trigonometrical Expressions, has been regulated solely by the principles of utility; and such of the forms as are the most important, and the most frequently quoted, are distinguished by an asterisk. Those which relate specially to Spherical Trigonometry, have been demonstrated in the Second Part of the preceding Treatise; and of the rest, the greater number will either be found amongst the results of the Introduction to that Part, or may very easily be derived from them.

TABLE I.

Forms, exhibiting the Values of the several Functions of a Circular Arch (A.)

[The Radius being *Unity.*]

Values of sin A.

(1)......$\sqrt{(1 - \cos^2 A)}$.*

(2)......$\sqrt{(\overline{1 + \cos A}.\ \overline{1 - \cos A})}$.

(3)......$\cos A \tan A$.*

(4)......$\dfrac{\cos A}{\cot A}$.

(5)......$\dfrac{1}{\operatorname{cosec} A}$.

(6)......$\dfrac{1}{\sqrt{(1 + \cot^2 A)}}$.

(7)......$\dfrac{\tan A}{\sqrt{(1 + \tan^2 A)}}$.

(8)......$2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A$.

(9)......$\sqrt{\dfrac{1 - \cos 2A}{2}}$.

(10)......$\dfrac{2 \tan \frac{1}{2} A}{1 + \tan^2 \frac{1}{2} A}$.

(11)......$\dfrac{2}{\cot \frac{1}{2} A + \tan \frac{1}{2} A}$.

(12)......$\dfrac{1}{\cot A + \tan \frac{1}{2} A}$.

(13)......$2 \sin^2 (45^\circ + \tfrac{1}{2} A) - 1$.

(14)......$1 - 2 \sin^2 (45^\circ - \tfrac{1}{2} A)$.

(15)......$\dfrac{1 - \tan^2 (45^\circ - \frac{1}{2} A)}{1 + \tan^2 (45^\circ - \frac{1}{2} A)}$.

(16)......$\dfrac{\tan (45^\circ + \frac{1}{2} A) - \tan (45^\circ - \frac{1}{2} A)}{\tan (45^\circ + \frac{1}{2} A) + \tan (45^\circ - \frac{1}{2} A)}$.

(17)......$\sin (60^\circ + A) - \sin (60^\circ - A)$.

Values of cos A.

(1)...... $\sqrt{(1-\sin^2 A)}$.*

(2)...... $\sqrt{(1+\sin A.\ \overline{1-\sin A})}$.

(3)...... $\dfrac{\sin A}{\tan A}$*.

(4)...... $\sin A \cot A$.

(5)...... $\dfrac{1}{\sec A}$.

(6)...... $\dfrac{1}{\sqrt{(1+\tan^2 A)}}$.

(7)...... $\dfrac{\cot A}{\sqrt{(1+\cot^2 A)}}$.

(8)...... $\cos^2 \frac{1}{2} A - \sin^2 \frac{1}{2} A$.

(9)...... $1 - 2 \sin^2 \frac{1}{2} A$.

(10)...... $2 \cos^2 \frac{1}{2} A - 1$.

(11)...... $\sqrt{\dfrac{1+\cos 2A}{2}}$.

(12)...... $\dfrac{1-\tan^2 \frac{1}{2} A}{1+\tan^2 \frac{1}{2} A}$.

(13)...... $\dfrac{\cot \frac{1}{2} A - \tan \frac{1}{2} A}{\cot \frac{1}{2} A + \tan \frac{1}{2} A}$.

(14)...... $\dfrac{1}{1+\tan A.\tan \frac{1}{2} A}$.

(15)...... $\dfrac{2}{\tan(45^\circ+\frac{1}{2}A)+\cot(45^\circ+\frac{1}{2}A)}$.

(16)...... $2 \cos(45^\circ+\frac{1}{2}A) \cos(45^\circ \sim \frac{1}{2}A)$.

(17)...... $\cos(60^\circ+A) \cos(60^\circ \sim A)$.

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† See Note (A) at the end of the Tables.

Values of tan A.

(1)...... $\sqrt{(\sec^2 A - 1)}$.*

(2)...... $\sqrt{(\sec A + 1) \cdot (\sec A - 1)}$.

(3)...... $\frac{\sin A}{\cos A}$.*

(4)...... $\frac{1}{\cot A}$.*

(5)...... $\sqrt{\left(\frac{1}{\cos^2 A} - 1\right)}$.

(6)...... $\frac{\sin A}{\sqrt{(1 - \sin^2 A)}}$.

(7)...... $\frac{\sqrt{(1 - \cos^2 A)}}{\cos A}$.

(8)...... $\frac{2 \tan \frac{1}{2} A}{1 - \tan^2 \frac{1}{2} A}$.

(9)...... $\frac{2 \cot \frac{1}{2} A}{\cot^2 \frac{1}{2} A - 1}$.

(10)...... $\frac{2}{\cot \frac{1}{2} A - \tan \frac{1}{2} A}$.

(11)...... $\cot A - 2 \cot 2 A$.

(12)...... $\frac{1 - \cos 2 A}{\sin 2 A}$.

(13)...... $\frac{\sin 2 A}{1 + \cos 2 A}$.

(14)...... $\sqrt{\frac{1 - \cos 2 A}{1 + \cos 2 A}}$.

(15)...... $\frac{\tan (45^\circ + \frac{1}{2} A) - \tan (45^\circ - \frac{1}{2} A)}{2}$.

Values of sec A.

(1)...... $\sqrt{(\tan^2 A + 1)}$.*

(2...... $\dfrac{1}{\cos A}$.†

(3)...... $\dfrac{\tan A}{\sin A}$.

(4)......$\cot\text{an}\ \frac{1}{2}(90^\circ - A) - \tan A$.*

Values of vers. sin A.

(1)......$1 - \cos A$.

(2)......$2 \sin^2 \frac{1}{2} A$.

Values of cotang A.

(1)...... $\sqrt{(\text{cosec}^2 A - 1)}$.

(2)...... $\dfrac{1}{\tan A}$.*

(3)...... $\dfrac{\cos A}{\sin A}$.*

Values of cosec A.

(1)...... $\sqrt{(\cot\text{an}^2 A + 1)}$.

(2)...... $\dfrac{1}{\sin A}$.

TABLE II.

Forms, involving Functions of two or more Circular Arches, A, B, C, &c.

[The Radius being *Unity*.]

(1)...$\tan A \cot A = \tan B \cot B$.

(2)...$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$.*

(3)...$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$.*

(4)...$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$.*

(5)...$\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$.

$= \cos^2 B - \cos^2 A$.

$= (\sin A + \sin B) \,.\, (\sin A - \sin B)$.

$= (\cos B + \cos A) \,.\, (\cos B - \cos A)$.

(6)...$\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$.

$= (\cos A + \sin B) \,.\, (\cos A - \sin B)$.

(7)...$\sin(45^\circ \pm B) = \cos(45^\circ \mp B)$.

$= \dfrac{\cos B \pm \sin B}{\sqrt{2}}$.

(8)...$\tan(45^\circ \pm B) = \dfrac{1 \pm \tan B}{1 \mp \tan B}$.

(9)...$\tan^2(45^\circ \pm \frac{1}{2} B) = \dfrac{1 \pm \sin B}{1 \mp \sin B}$.

(10)...$\dfrac{\sin(A+B)}{\sin(A-B)} = \dfrac{\tan A + \tan B}{\tan A - \tan B}$.

$= \dfrac{\cot B + \cot A}{\cot B - \cot A}$.

(11)...$\dfrac{\cos(A+B)}{\cos(A \sim B)} = \dfrac{\cot B - \tan A}{\cot B + \tan A}$.

$= \dfrac{\cot A - \tan B}{\cot A + \tan B}$.

(12)...$\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}$.

(13)...$\dfrac{\cos B+\cos A}{\cos B-\cos A}=\dfrac{\cot\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$.

(14)...$\sin A\cos B=\frac{1}{2}\sin(A+B)+\frac{1}{2}\sin(A-B)$.

(15)...$\cos A\sin B=\frac{1}{2}\sin(A+B)-\frac{1}{2}\sin(A-B)$.

(16)...$\sin A\sin B=\frac{1}{2}\cos(A\sim B)-\frac{1}{2}\cos(A+B)$.

(17)...$\cos A\cos B=\frac{1}{2}\cos(A+B)+\frac{1}{2}\cos(A\sim B)$.

(18)...$\sin A+\sin B=2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A\sim B)$.

(19)...$\sin A-\sin B=2\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B)$.

(20)...$\cos A+\cos B=2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A\sim B)$.

(21)...$\cos B-\cos A=2\sin\frac{1}{2}(A-B)\sin\frac{1}{2}(A+B)$.

(22)...$\tan A\pm\tan B=\dfrac{\sin(A\pm B)}{\cos A\cos B}$.

(23)...$\cot A\pm\cot B=\dfrac{\sin(A\pm B)}{\sin A\sin B}$.

(24)...$\tan^2 A-\tan^2 B=\dfrac{\sin(A-B)\sin(A+B)}{\cos^2 A\cos^2 B}$.

(25)...$\cot^2 B-\cot^2 A=\dfrac{\sin(A-B)\sin(A+B)}{\sin^2 A\sin^2 B}$.

(26)...$\left.\begin{array}{l}\sin A\sin(B-C)+\sin B\sin(C-A)\\ +\sin C\sin(A-B)\end{array}\right\}=0.$

(27)...$\left.\begin{array}{l}\cos A\sin(B-C)+\cos B\sin(C-A)\\ +\cos C\sin(A-B)\end{array}\right\}=0.$

If $B=\dfrac{360^\circ}{n}$,

(28)...$\sin(A+B)+\sin(A+2B)+\&c.+\sin(A+nB)=0.$

(29)...$\cos(A+B)+\cos(A+2B)+\&c.+\cos(A+nB)=0.$

If $A+B+C=180^\circ$,

(30)...$\tan A+\tan B+\tan C=\tan A\tan B\tan C$.

If $A+B+C=n\,.\,180^\circ$,

(31)...$\tan(A+B)+\tan C=0$.

If $A+B+C=(2n+1)\,90^\circ$,

(32)...$\cot A+\cot B+\cot C=\cot A\cot B\cot C$.

TABLE III.

Values of the Functions of Parts and Multiples of a Circular Arch (A).

[The Radius being *Unity*.]

(1)...$\sin\frac{1}{2}A = \sqrt{\frac{1-\cos A}{2}}$.*

(2)...$\cos\frac{1}{2}A = \sqrt{\frac{1+\cos A}{2}}$.

(3)...$\tan\frac{1}{2}A = \frac{1-\cos A}{\sin A}$.

$= \frac{\sin A}{1+\cos A}$.

$= \sqrt{\frac{1-\cos A}{1+\cos A}}$.

$= \frac{\sqrt{(1+\tan^2 A)}-1}{\tan A}$.

(4)...$\cos mA + \sqrt{-1}\,.\,\sin mA = (\cos A + \sqrt{-1}\,.\,\sin A)^m$.*

(5)...$\sin 2A = 2\sin A\sqrt{(1-\sin^2 A)} = 2\cos A\sqrt{(1-\cos^2 A)}$.

$\sin 3A = 3\sin A - 4\sin^3 A = (4\cos^2 A - 1)\,.\,\sqrt{(1-\cos^2 A)}$.

&c. = &c.

$$\sin mA = m\cos^{m-1}A\sin A - m.\frac{m-1}{2}.\frac{m-2}{3}\cos^{m-3}A\sin^3 A + \&c.$$

$$= \cos A\left[\,m\sin A - \frac{m(m^2-4)}{2.3}\sin^3 A + m\,.\,\frac{(m^2-4).(m^2-16)}{2.3.4.5}\right.$$

$$\left.\sin^5 A - \&c.\right] \quad (m \text{ being } \textit{even}.)$$

$$= m\sin A - \frac{m.(m^2-1)}{2.3}\sin^3 A + \frac{m(m^2-1)(m^2-9)}{2.3.4.5}\sin^5 A - \&c.$$

(m being *odd*).

(6)...$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A.$

$\cos 3A = 4\cos^3 A - 3\cos A = (1 - 4\sin^2 A) \cdot \sqrt{(1 - \sin^2 A)}.$

&c. = &c.

$$\cos mA = \cos^m A - m\frac{m-1}{2}\cos^{m-2}A\sin^2 A + m\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot\frac{m-3}{4}\cos^{m-4}A\sin^4 A - \&c$$

$$= 2^{m-1}\cos^m A - 2^{m-3}m\cos^{m-2}A + 2^{m-5}m.\frac{m-3}{2}\cos^{m-4}A \cdot 2^{m-7}\frac{m(m-4)(m-5)}{2.3}\cos^{m-6}A + \&c.$$

(7)...$\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}.$

$\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$

$\tan 4A = \dfrac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}.$

&c. = &c.

$$\tan mA = \frac{m\tan A - m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\tan^3 A + \&c.}{1 - m\cdot\frac{m-1}{2}\tan^2 A + \&c.}.$$

(8)...$\cot 2A = \dfrac{1 - \tan^2 A}{2\tan A}.$

(9)...$\sec 2A = \dfrac{\sec^2 A}{1 - \tan^2 A} = \dfrac{1 + \tan^2 A}{1 - \tan^2 A}.$

(10)...$\operatorname{cosec} 2A = \dfrac{\sec^2 A}{2\tan A} = \dfrac{1 + \tan^2 A}{2\tan A}.$

TABLE IV.

Values of the Powers of the Sine, and of the Cosine of a Circular Arch (A).

[The Radius being *Unity*, and π being put for the Semi-circumference.]

(1)...$2\sin^2 A = 1 - \cos 2A$.

(2)...$4\sin^3 A = 3\sin A - \sin 3A$.

(3)...$8\sin^4 A = 3 - 4\cos 2A + \cos 4A$.

&c. = &c.

(4)...$2^{m-1}\sin^m A =$

$$\left.\begin{cases}\cos m\frac{\pi}{2}\,.\,(\cos mA + m\cos(m-2)A + m\,.\,\frac{m-1}{2}\cos(m-4)A + \&c.) \\ +\sin m\frac{\pi}{2}\,.\,(\sin mA + m\sin(m-2)A + m\,.\,\frac{m-1}{2}\sin(m-4)A + \&c.)\end{cases}\right\}(\dagger)$$

(5)...$2\cos^2 A = 1 + \cos 2A$.

(6)...$4\cos^3 A = 3\cos A + \cos 3A$.

(7)...$8\cos^4 A = 3 + 4\cos 2A + \cos 4A$.

&c. = &c.

(8)...$2^{m-1}\cos^m A = \cos mA + m\cos(m-2)A + m\,.\,\frac{m-1}{2}\cos(m-4)A + \&c.$*

TABLE V.

Values of a Circular Arch (A), *in terms of the Functions of the Arch; and the Reverse Expressions.*

[The Radius being *Unity*, and π the Semi-circumference.]

(1)...$A = \sin A + \frac{\sin^3 A}{2.3} + \frac{3\sin^5 A}{2.4.5} + \frac{3.5\sin^7 A}{2.4.6.7} + \frac{3.5.7\sin^9 A}{2.4.6.8.9} + \&c.$*

(2)...$A = \frac{\pi}{2} - \cos A - \frac{\cos^3 A}{2.3} - \frac{3\cos^5 A}{2.4.5} - \frac{3.5\cos^7 A}{2.4.6.7} - \&c.$

(3)...$A = \tan A - \frac{1}{3}\tan^3 A + \frac{1}{5}\tan^5 A - \frac{1}{7}\tan^7 A + \&c.$*

(4)...$A = \frac{\pi}{2} - \cot A + \frac{1}{3}\cot^3 A - \frac{1}{5}\cot^5 A + \frac{1}{7}\cot^7 A - \&c.$

† See note (B) at the end of the Tables.

$$(5)\ldots\sin A = A - \frac{A^3}{2.3} + \frac{A^5}{2.3.4.5} - \frac{A^7}{2.3.4.5.6.7} + \&c.^*$$

$$(6)\ldots\cos A = 1 - \frac{A^2}{2} + \frac{A^4}{2.3.4} - \frac{A^6}{2.3.4.5.6} + \&c.$$

$$(7)\ldots\tan A = A + \frac{A^3}{3} + \frac{2A^5}{3.5} + \frac{17A^7}{3.5.7.3} + \frac{62A^9}{3.5.7.9.3} + \&c.^*$$

$$(8)\ldots\cot A = \frac{1}{A} - \frac{A}{3} - \frac{A^3}{3^2.5} - \frac{2A^5}{3^3.5.7} - \frac{A^7}{3^3.5^2.7} - \frac{2A^9}{3^3.5.7.9.11}.$$

TABLE VI.

Forms applicable to the Solution of Plane Triangles.

In this and in the following Tables A, A′, A″, *are put for the Angles of a Triangle;* S, S′, S″ *for the opposite Sides;* P *for its Semi-perimeter;* T *for its Surface;* D *and* D′ *for the Segments into which any Side, as* S″, *is divided by the Perpendicular let fall upon it from the opposite Angle* A″; V *and* V′ *for the Segments of the Angle* A″, *opposite to* D *and* D′ *respectively; and Unity for the Tabular Radius.*

CASE I.

Let one of the Angles, as A, be a Right Angle.

$$(1)\ldots S = \sqrt{(S')^2 + (S'')^2}.^*$$

$$S' = \sqrt{[(S+S'').(S-S'')]}.^*$$

$$S'' = \sqrt{[(S+S').(S-S')]}.$$

$$(2)\ldots\frac{S'}{S} = \sin A' = \cos A''.^*$$

$$\frac{S''}{S} = \sin A'' = \cos A'.$$

$$(3)\ldots\frac{S'}{S''} = \tan A' = \cot A''.^*$$

$$\frac{S''}{S'} = \tan A'' = \cot A'.$$

CASE II.

Let none of the Angles be a Right Angle.

$$(4)\ldots\frac{S}{S'} = \frac{\sin A}{\sin A'}.^* \ (\dagger)$$

$$(5)\ldots\sin A = \frac{2\sqrt{[P.(P-S).(P-S').(P-S'')]}}{S'\,S''}.^*$$

$$(6)\ldots\cos A = \frac{(S')^2 + (S'')^2 - S^2}{2\,S'\,S''}.^*$$

† See Note (C) at the end of the Tables.

(7)...$\tan A = \sqrt{\left[\left(\frac{2\, S'S''}{(S')^2+(S'')^2-S^2}\right)^2-1\right]}$

(8)...$\sin^2 \frac{1}{2} A = \frac{(P-S').(P-S'')}{S'\, S''}$.*

(9)...$\cos^2 \frac{1}{2} A = \frac{P.(P-S)}{S'.\, S''}$.

(10)...$\tan^2 \frac{1}{2} A = \frac{(P-S').(P-S'')}{P.(P-S)}$.

(11)...$\frac{S''}{S+S'} = \frac{S \sim S'}{D \pm D'}$.*

(12)...$\frac{\sin V}{\sin V'} = \frac{\cos A}{\cos A'}$.

(13)...$\frac{D}{D'} = \frac{\tan A'}{\tan A}$.

(14)...$\frac{S+S'}{S \sim S'} = \frac{\tan \frac{1}{2}(A+A')}{\tan \frac{1}{2}(A \sim A')}$.*

(15)...$S = \sqrt{[(S')^2+(S'')^2-2\, S'\, S'' \cos A]}$.*

$$\tan x = 2 \sin \tfrac{1}{2} A . \frac{\sqrt{(S'.\, S'')}}{S' \sim S''}.$$

$$S = \frac{S' \sim S''}{\cos x}.$$

(16)...$S = \frac{S'}{\cos A'' + \sin A'' \cot A}$.

(17)...$S = (\cos A'' + \sin A'' \cot A') . S'$.

(18)...$\tan A = \frac{S. \sin A'}{\pm\sqrt{[(S')^2 - \sin^2 A'.S^2]}}$.

(19)...$\tan A = \frac{S . \sin A'}{S'' - \cos A'. S}$.

(20)...$S = \sqrt{\frac{2\, T . \sin A}{\sin A' \sin A''}}$.

(21)...$S = \sqrt{[(S')^2+(S'')^2 \pm 2\sqrt{((S')^2 (S'')^2 - 4\, T^2)}]}$.

TABLE VII.

Forms applicable to the Solution of Spherical Triangles: in which, besides the Notation already described, p *is put for the Semi-sum of the three Angles of a Spherical Triangle.*

Case I.

Let one, and only one, Angle, as A, be a Right Angle.

(1)... $\sin A' = \dfrac{\sin S'}{\sin S}$;

$\sin A'' = \dfrac{\sin S''}{\sin S}$. *

(2)... $\cos A' = \dfrac{\tan S''}{\tan S}$;

$\cos A'' = \dfrac{\tan S'}{\tan S}$. *

(3)... $\sin A' = \dfrac{\cos A''}{\cos S''}$;

$\sin A'' = \dfrac{\cos A'}{\cos S'}$.

(4)... $\cos S' = \dfrac{\cos S}{\cos S''}$;

$\cos S'' = \dfrac{\cos S}{\cos S'}$.

(5)... $\sin S' = \dfrac{\tan S''}{\tan A''}$;

$\sin S'' = \dfrac{\tan S'}{\tan A'}$.

(6)... $\cos S = \dfrac{\cot A''}{\tan A'}$

$= \dfrac{1}{\tan A' \tan A''}$.

(7)... $\tan^2 \frac{1}{2} S = \dfrac{-\cos (A' + A'')}{\cos (A' \sim A'')}$.

(8)... $\tan^2 \frac{1}{2} S'' = \tan \dfrac{A' + A'' - 90^\circ}{2} \cot \dfrac{A' - A'' + 90^\circ}{2}$.

(9)... $\tan^2 \frac{1}{2} A'' = \dfrac{\sin (S \sim S')}{\sin (S + S')}$.

(10)... $\tan^2 \frac{1}{2} S = \tan \frac{1}{2} (S + S') \tan \frac{1}{2} (S \sim S')$.

Case II.

Let none of the Angles be a Right Angle.

(11)... $\dfrac{\sin S}{\sin S'} = \dfrac{\sin A}{\sin A'}$. *(†)

(12)... $\cos S = \cos S' \cos S'' + \sin S' \sin S'' \cos A$. *

(13)... $\cos A = -\cos A' \cos A'' + \sin A' \sin A'' \cos S$. *

(14)... $\cos S \cos A' = \cot S'' \sin S - \sin A' \cot A''$. *

† See Note (D) at the end of the Tables.

(15)...$\tan D = \cos A \tan S'$.

(16)...$\cot V = \tan A \cos S'$.

(17)...$\dfrac{\cos D}{\cos D'} = \dfrac{\cos S'}{\cos S}$.*

(18)...$\dfrac{\sin V}{\sin V'} = \dfrac{\cos A}{\cos A'}$.

(19)...$\dfrac{\sin D}{\sin D'} = \dfrac{\tan A'}{\tan A}$.*

(20)...$\dfrac{\cos V}{\cos V'} = \dfrac{\tan S}{\tan S'}$.*

(21)...$\tan \frac{1}{2}(D-D') = \tan \frac{1}{2}(S'-S) \tan \frac{1}{2}(S'+S) \cot \frac{1}{2} S''$.

(22)...$\tan \frac{1}{2}(V-V') = \tan \frac{1}{2}(A'-A) \tan \frac{1}{2}(A'+A) \tan \frac{1}{2} A''$.

(23)...$\sin^2 \frac{1}{2} A = \dfrac{\sin(P-S')\sin(P-S'')}{\sin S' \sin S''}$.*

(24)...$\cos^2 \frac{1}{2} A = \dfrac{\sin(P-S)\sin P}{\sin S' \sin S''}$.*

(25)...$\tan^2 \frac{1}{2} A = \dfrac{\sin(P-S')\sin(P-S'')}{\sin P \sin(P-S)}$.

(26)...$\sin^2 \frac{1}{2} S = \dfrac{-\cos p \cos(p-A)}{\sin A' \sin A''}$*.

(27)...$\cos^2 \frac{1}{2} S = \dfrac{\cos(p-A')\cos(p-A'')}{\sin A' \sin A''}$.*

(28)...$\tan^2 \frac{1}{2} S = \dfrac{-\cos p \cos(p-A)}{\cos(p-A')\cos(p-A'')}$.

(29)...$\dfrac{\tan \frac{1}{2}(S+S')}{\tan \frac{1}{2}(S-S')} = \dfrac{\tan \frac{1}{2}(A+A')}{\tan \frac{1}{2}(A-A')}$.

(30)...$\tan \frac{1}{2}(A'-A) = \dfrac{\sin \frac{1}{2}(S'-S)}{\sin \frac{1}{2}(S'+S)} \cot \frac{1}{2} A''$.*

$\tan \frac{1}{2}(A'+A) = \dfrac{\cos \frac{1}{2}(S'-S)}{\cos \frac{1}{2}(S'+S)} \cot \frac{1}{2} A''$.*

(31)...$\tan \frac{1}{2}(S'-S) = \dfrac{\sin \frac{1}{2}(A'-A)}{\sin \frac{1}{2}(A'+A)} \tan \frac{1}{2} S''$.*

(32)...$\tan \frac{1}{2}(S'+S) = \dfrac{\cos \frac{1}{2}(A'-A)}{\cos \frac{1}{2}(A'+A)} \tan \frac{1}{2} S''$.*(†)

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† See Note (E) at the end of the Tables.

TABLE VIII.

Values of the Surfaces of certain Plane and Solid Figures: in which, besides the Notation already described, c *is put for the Circumference, and* r *for the Radius, of a great Circle in a Sphere;* π *also is put for* $\frac{355}{113}$ (†) *the Ratio of the Circumference of a Circle to its Diameter.*

I. The Surface of a Plane Triangle.

(1)...$\frac{1}{2}\, S'\, S''\, \sin A$.*

(2)...$\frac{1}{2}\, S'' \sin A' \left(S'' \cos A + \surd\,[(S')^2 - (S'')^2 \sin^2 A]\right)$.

(3)...$\surd\,[P.\,(P-S).\,(P-S')\,(P-S'')]$.*

(4)...$\frac{\sin A' \sin A''}{\sin A} \frac{1}{2}\, S^2$.

II. The Surface of a Parallelogram.

$S\,.\,S' \sin A$.*

III. The Surface of a Trapezium,

(of which two Sides, S and S'', are parallel.)

$$\frac{1}{2}\,\frac{S+S''}{S-S''}\surd\left[2\left((S')^2+(S''')^2\right).(S-S'')^2-(S-S'')^4-\left((S')^4-(S''')^4\right)\right].$$

IV. The Surface of a Quadrilateral Plane Figure,

(of which one Angle, contained by the Sides S and S', is a Right Angle.)

$$\frac{1}{2}S.S'+\surd\,[4\,(P-S)(P-S')(P-S'')(P-S''')-S.S'(S.S'+2S''S''')].$$

V. The Surface of a Quadrilateral Figure inscriptible in a Circle.

$$\surd\,[\,(P-S)\,(P-S')\,(P-S'')\,(P-S''')].$$

VI. The Surface of a Rhombus,

(of which the Diameters are D and D').

$\frac{1}{2}\, D.\, D'$.

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† See note (F) at the end of the Tables.

VII. The Surface of a Regular Polygon, having n Sides.

$$\tfrac{1}{4}\, n\, S^2 . \cot \frac{180^\circ}{n} .*$$

VIII. A Circle,

(of which r is the Radius, and c the Circumference.)

(1)...... $\tfrac{1}{2}\, c\, r$.

(2)...... $\pi . r^2$. i. e. $3.1415926535\, r^2$*.

(3)...... $\dfrac{c^2}{4\pi}$.

IX. An Ellipse,

(of which a and b are the Semi-axes, and d the Eccentricity.)

(1)...... $a\, b\, \pi$ *

(2)...... $a \surd (a^2 - d^2) . \pi$.

(3)...... $b \surd (b^2 + d^2) . \pi$.

X. The Surface of a Sphere.

(1)...... $2\, c\, r$.

(2)...... $4\, r^2\, \pi$.*

(3)...... $\dfrac{c^2}{\pi}$.

XI. The convex Surface of a Segment of a Sphere,

(l being the Axis, and B the Spherical Polar Distance of the Base of the Segment.)

(1)...... $l . c$.

(2)...... $l . 2\, r\, \pi$.

(3)...... $4\, r^2 \pi . \sin^2 \tfrac{1}{2} B$ *.

XII. The Surface of a Spherical Zone.

(l being the Axis, and B, B' the Spherical Polar Distances of its terminating Circles.)

(1)...... $l . c$.

(2)...... $l . 2\, r\, \pi$.

(3)...... $4\, r^2 \pi \sin \tfrac{1}{2} (B' - B) \sin \tfrac{1}{2} (B' + B)$* (†)

† See Note (G) at the end of the Tables.

XIII. The Convex Surface of a Spherical Sector,

(A° being the Angle of inclination of the two terminating Semicircles.)

$$(1)\ldots\ldots c\, r.\frac{A^{\circ}}{180}.$$

$$(2)\ldots\ldots 2\, r^2 \pi.\frac{A^{\circ}}{180}.$$

XIV. The Surface of a Spherical Triangle.

$$(1)\ldots\ldots\frac{A+A'+A''-180}{180}.r^2\pi.$$

$$(2)\ldots\ldots(A+A'+A''-180).\, r^2 \sin 1''.*$$

XV. The Surface of a Spherical Quadrilateral Figure (Q), (of which two opposite sides, H, H', are perpendicular to the Base B.)

$$\tan\tfrac{1}{2}Q=\frac{\sin\frac{1}{2}(H'+H)}{\sin\frac{1}{2}(H'-H)}.\tan\tfrac{1}{2}B.$$

TABLE IX.

Values of the Solid Contents of certain Figures. (†)

I. The Solid Content of a Cylinder,

$(1)\ldots\ldots\frac{1}{2}c\, r\, h.$

$(2)\ldots\ldots\pi.\, r^2 h$ or $3.14159 .\, r^2 h.$

II. The Solid Content of a Sphere.

$\frac{4\pi r^3}{3}$ i. e. $4.18878.\, r^3.$

III. The Solid Content of a Spheroid,

$\frac{1}{6}\pi a b^2$ or $.523598\, a b^2.$

IV. The Solid Content of a Cone.

$\frac{1}{3}\pi r^2 h$ or $1.047197.\, r^2 h.$

V. The Solid Content of a Frustum of a Cone.

$$\tfrac{1}{12}.\pi.\frac{D^3-d^3}{D-d}.h \quad\text{or}\quad .261799.\frac{D^3-d^3}{D-d}.h.$$

VI. The Solid Content of the Frustum of a Pyramid.

$$\tfrac{1}{3}.(A+a+\sqrt{Aa}.).h.$$

† See Note (H) at the end of the Tables.

NOTES.

(A). If *Unity* be divided by the several values of the cosine of A, a series of values of sec A will be obtained. By inverting, likewise, the several values of tan A, a series of values may be found of cot A; and, by inverting the forms which express the sine of A, there will result forms, which express, each of them, a value of cosec A.

(B). If m be *even*, $\sin\left(m\frac{\pi}{2}\right)=0$; and if m be *odd*, $\cos\left(m\frac{\pi}{2}\right)=0$: Again, if m be even and of the form $2.^{2n}\,p$, in which p is an odd number, then $\cos\left(m\frac{\pi}{2}\right)=+1$; and the terms of the series are alternately positive and negative, the first term being positive: but if m be *even*, and of the form $2\,p$, then $\cos\left(m\frac{\pi}{2}\right) = -1$; and the terms are alternately negative and positive.

Lastly, if m be *odd*, the signs of the terms of the series, which expresses the value of $\sin{}^mA$, will be alternately positive and negative, the first of them being positive; when m is any of the numbers in the series 1, 5, 9, &c. Otherwise, the terms will be alternately negative and positive.

(C). The subsequent forms in this Table are all of them *general;* although only one particular expression, under each of them, is here given, because of the extreme facility of deducing from it the rest. With the system of Notation, which has been adopted, the rule for making such a deduction is this: "Place an additional accent over every letter of the form, which has not two accents; and wherever there are two accents suppress them." Thus, from $\frac{S}{S'} = \frac{\sin A}{\sin A'}$ are derived $\frac{S'}{S''} = \frac{\sin A'}{\sin A''}$; and $\frac{S''}{S} = \frac{\sin A''}{\sin A}$.

(D). It is left to the reader to adapt this and the following Forms of Table VII. to the different combinations of the sides and the angles of a spherical triangle, to which they are respectively applicable, by means of the rule given in the preceding note.

(E). An inspection of Table VII. will suffice to suggest to the reader the particular cases in which the several expressions contained in it are to be used. It is manifest, for example, that Forms 15, 16 serve for the solution of an oblique-angled spherical triangle, whenever an angle, and a side adjacent to it, are two of the data. When the three sides are given, Forms 21 or 23, and 24 or 25, may be employed, as may Forms 22 or 26, and 27 or 28, when the three angles are given; and if there be given two sides and the included angle, or a side and the two angles adjacent to it, then may Forms 29 and 30 be used with great conveniency.

(F). This ratio may be derived from the following rule, which is easy to recollect: "Write down the first three of the *odd* numbers, each twice, and place the mark of proportion between the two middle digits." Thus, 113 : 355, is the ratio of the diameter of a circle to the circumference.

(G). By the help of Forms XI. 3. XII. 3. Tab. VIII. the quantity of surface in the frozen, the temperate, and the torrid zones, may readily be computed, if the Earth be considered as a sphere.

(H). In the Forms of Table IX. *h* is every where put for the height of the figure; *c* for the circumference of the cylinder; *r* for the radius of the base of the cylinder, and of the base of the cone, and also for the sphere's radius; *a* for the axis about which the spheroid is generated, and *b* for its other axis; *D*, and *d*, for the diameters of the ends of the frustum of the cone; and *A*, *a*, for the surfaces of the ends of the pyramid; the cone and pyramid being, each of them, supposed to be cut by a plane parallel to its axis.

It is evident that some of the Forms of this Table are applicable to the measurement of timber. Alhough they do not, properly, belong to the subject of Trigonometry, they are here given, on account of their usefulness.

THE END.